An Isomorphism Theorem for Henselian Algebraic Extensions of Valued Fields

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In general, the value groups and the residue fields do not suffice to classify the algebraic henselian extensions of a valued field $\mathbf{K}$, up to isomorphism over $\mathbf{K}$. We define a stronger, yet natural structure which carries information about additive and multiplicative congruences in the valued field, extending the information carried by value groups and residue fields. We discuss the cases where these "mixed structures" give a solution of the classification problem.

1 Introduction

Consider the following problem: given a valued field $\mathbf{K} = (K, v)$, find elementary invariants to classify, up to isomorphism over $\mathbf{K}$, the algebraic henselian valued field extensions of $\mathbf{K}$.

In an important special case, an answer to the question above is given by the following result due to Prestel and Roquette ([3], Corollary 3.11):

Theorem 1.1 *Let $\mathbf{K} = (K, v)$ be a p-valued field and $\mathbf{L} = (L, v)$, $\mathbf{F} = (F, v)$ be two henselian p-valued algebraic extensions of the same p-rank as $\mathbf{K}$. The necessary and sufficient condition for $\mathbf{L}$, $\mathbf{F}$ to be $\mathbf{K}$–isomorphic (as valued fields) is that $K \cap L^n = K \cap F^n$ for each $n \in I\!N$.*

The main goal of this paper is to suitably extend Theorem 1.1 to a more general case including that of valued fields of characteristic zero.

Given a valued field $\mathbf{K} = (K, v)$ let us denote by $\mathcal{O}_{\mathbf{K}}$ the valuation ring, by $\overline{K}$ the residue field and by vK the value group $vK = \{va \mid 0 \neq a \in K\}$. Let

$vK^{\geq 0} := \{\delta \in vK \mid \delta \geq 0\}$. If there is danger of confusion, we will write $v_{\mathbf{K}}$ to denote to which field the valuation refers.

For $\delta \in vK^{\geq 0}$, let $\mathcal{M}_{\mathbf{K}}^{\delta}$ be the ideal $\{a \in \mathcal{O}_{\mathbf{K}} : va > \delta\}$ of $\mathcal{O}_{\mathbf{K}}$. In particular, $\mathcal{M}_{\mathbf{K}}^{0} = \mathcal{M}_{\mathbf{K}}$ is the maximal ideal of the valuation ring $\mathcal{O}_{\mathbf{K}}$. Denote by $\mathcal{O}_{\mathbf{K}}^{\delta}$ the factor ring $\mathcal{O}_{\mathbf{K}}/\mathcal{M}_{\mathbf{K}}^{\delta}$; this is a local ring with maximal ideal $\mathcal{M}_{\mathbf{K}}/\mathcal{M}_{\mathbf{K}}^{\delta}$. In particular, $\mathcal{O}_{\mathbf{K}}^{0} = \overline{K}$. We write π_δ for the canonical projection $\mathcal{O}_{\mathbf{K}} \longrightarrow \mathcal{O}_{\mathbf{K}}^{\delta}$. Note that for $a \in \mathcal{O}_{\mathbf{K}}$, the projection $\pi_\delta a$ is an invertible element of $\mathcal{O}_{\mathbf{K}}^{\delta}$ if and only if $va = 0$.

On the other hand, consider the multiplicative groups $G_{\mathbf{K}}^{\delta} = K^\times / 1 + \mathcal{M}_{\mathbf{K}}^{\delta}$ for $\delta \in vK^{\geq 0}$. In particular,

$$G_{\mathbf{K}} := G_{\mathbf{K}}^{0} = K^\times / 1 + \mathcal{M}_{\mathbf{K}}\,.$$

We write π_δ^* for the canonical projection $K^\times \longrightarrow G_{\mathbf{K}}^{\delta}$. Note that $G_{\mathbf{K}}^{\delta}$ is the group of multiplicative congruence classes modulo $\mathcal{M}_{\mathbf{K}}^{\delta}$ in the sense of Hasse.

Given $\delta \in vK^{\geq 0}$, the local ring $\mathcal{O}_{\mathbf{K}}^{2\delta}$ and the group $G_{\mathbf{K}}^{\delta}$ are related through the map Θ_δ of the subset $\{x \in \mathcal{O}_{\mathbf{K}}^{2\delta} \mid x^2 \neq 0\} \subset \mathcal{O}_{\mathbf{K}}^{2\delta}$ into $G_{\mathbf{K}}^{\delta}$ which is given by

$$\Theta_\delta(a + \mathcal{M}_{\mathbf{K}}^{2\delta}) = a(1 + \mathcal{M}_{\mathbf{K}}^{\delta}) \text{ for all } a \in \mathcal{O}_{\mathbf{K}} \text{ subject to } va \leq \delta;$$

note that this map is welldefined since for elements of value $\leq \delta$, additive congruence modulo $\mathcal{M}_{\mathbf{K}}^{2\delta}$ implies multiplicative congruence modulo $\mathcal{M}_{\mathbf{K}}^{\delta}$. We have $\pi_\delta^* a = \Theta_\delta \pi_{2\delta} a$ for all $a \in \mathcal{O}_{\mathbf{K}}$ subject to $va \leq \delta$.

For $\delta \in vK^{\geq 0}$, consider the system

$$\mathbf{K}_\delta = (\mathcal{O}_{\mathbf{K}}^{2\delta},\ G_{\mathbf{K}}^{\delta},\ \Theta_\delta)$$

and call it the *mixed δ-structure* assigned to $\mathbf{K}$. In particular, $\mathbf{K}_0$ is the pair $(\overline{K},\ G_K)$ together with the embedding

$$\overline{K}^\times \longrightarrow G_{\mathbf{K}}\,.$$

Let two extensions $\mathbf{L} = (L, v)$ and $\mathbf{F} = (F, v)$ of $\mathbf{K}$ be given. Then every $\mathbf{K}$–embedding of $\mathbf{L}$ into $\mathbf{F}$ induces a $\mathbf{K}_\delta$–embedding of $\mathbf{L}_\delta$ into $\mathbf{F}_\delta$ for each $\delta \in vK^{\geq 0}$. We want to ask for the converse, i.e. whether from the existence of embeddings of the mixed δ–structures, the existence of a $\mathbf{K}$–embedding of $\mathbf{L}$ into $\mathbf{F}$ may be deduced. However, we do not want to let δ run through all of $vK^{\geq 0}$; we will gain a better information if we restrict δ to $\Delta^{\geq 0} = \Delta \cap vK^{\geq 0}$ for some convex subgroup Δ of vK, compensating by a suitable hypothesis on the extension $\mathbf{L}$ of $\mathbf{K}$.

Given a convex subgroup Δ of vK, there is a unique coarsening v_Δ of the valuation v whose value group is isomorphic to vK/Δ and whose residue field Kv_Δ carries a valuation $\overline{v}_\Delta$ with value group isomorphic to Δ such that v is the composition of v_Δ and $\overline{v}_\Delta$; for details, see e.g. [5].

For every extension $\mathbf{L}$ of $\mathbf{K}$, there is a prolongation of v_Δ to L; it is uniquely determined in case $L|K$ is algebraic. If (L, v) is henselian, then so is (L, v_Δ); if in addition, $L|K$ is algebraic, then (L, v_Δ) may be viewed as an algebraic extension of some henselization of (K, v_Δ).

An algebraic extension $(k_1, w)|(k, w)$ of henselian fields is called *tame* if k_1 is contained in the *absolute ramification field* of (k, w) which is defined to be the ramification field of the extension $(k^{sep}, w)|(k, w)$, where k^{sep} denotes the separable–algebraic closure of k, carrying the unique prolongation of the henselian valuation w of k. An equivalent condition is the following: for every finite subextension $(k', w)|(k, w)$ of $(k_1, w)|(k, w)$, the following holds:

1) the residue field extension $k'w|kw$ is separable,
2) if $p = \mathrm{char}(kw) > 0$, then the ramification index $(wk' : wk)$ is prime to p,
3) the extension is defectless, i.e.

$$[k' : k] = [k'w : kw] \cdot (wk' : wk) \ . \tag{1}$$

Note that in general, the following *fundamental inequality* holds:

$$[k' : k] \geq [k'w : kw] \cdot (vk' : vk) \ . \tag{2}$$

With these prerequisites, we are able to formulate our main theorem:

Theorem 1.2 *Let $\mathbf{K} = (K, v)$ be a valued field, $\mathbf{L} = (L, v)$ an algebraic extension of $\mathbf{K}$ and $\mathbf{F} = (F, v)$ an arbitrary henselian extension of $\mathbf{K}$. If Δ is a convex subgroup of vK such that (L, v_Δ) is a tame extension of some henselization of (K, v_Δ), then the next statements are equivalent:*

i) $\mathbf{L}$ is $\mathbf{K}$–embeddable into $\mathbf{F}$,

ii) $\mathbf{L}_\delta$ is $\mathbf{K}_\delta$–embeddable into $\mathbf{F}_\delta$ for each $\delta \in \Delta^{\geq 0}$.

Note that the existence of such a convex subgroup Δ of vK implies that $L|K$ is separable; conversely, if $L|K$ is separable, then there is always such a Δ, namely $\Delta = vK$. With this choice, v_Δ is the trivial valuation, and $\overline{v}_\Delta$ may be identified with v.

Given two algebraic extensions L and F of K and K–embeddings $\sigma : L \to F$, $\sigma' : F \to L$, then both σ and σ' must be isomorphisms. Since every algebraic extension is the union of finite subextensions, this fact is simply seen by counting the number of roots in L and in F for every polynomial $f \in K[X]$. Hence, the following isomorphism theorem is an immediate consequence of Theorem 1.2.

Theorem 1.3 *Let $\mathbf{K} = (K, v)$ be a valued field and $\mathbf{L} = (L, v)$, $\mathbf{F} = (F, v)$ two henselian algebraic extensions of $\mathbf{K}$. If Δ is a convex subgroup of vK such that both (L, v_Δ) and (F, v_Δ) are tame extensions of some henselizations of (K, v_Δ), then the next statements are equivalent:*

i) $\mathbf{L}$ and $\mathbf{F}$ are $\mathbf{K}$–isomorphic,

ii) $\mathbf{L}_\delta$ and $\mathbf{F}_\delta$ are $\mathbf{K}_\delta$–isomorphic for each $\delta \in \Delta^{\geq 0}$.

Remarks.

a) In a canonical way, one may define projections $\pi_{\gamma,\delta} : \mathbf{K}_\gamma \longrightarrow \mathbf{K}_\delta$ for $\gamma \geq \delta$. Through such a projection, a $\mathbf{K}_\gamma$–embedding of $\mathbf{L}_\gamma$ into $\mathbf{F}_\gamma$ induces a $\mathbf{K}_\delta$–embedding of $\mathbf{L}_\delta$ into $\mathbf{F}_\delta$. This shows that in the above theorems, the range of δ may be restricted to an arbitrary cofinal subset of $\Delta^{\geq 0}$.

b) The map Θ_δ is always defined on the units of the ring $\mathcal{O}_{\mathbf{K}}^{2\delta}$, so one may be tempted to replace in our definition of mixed structures the map Θ_δ by its restriction to the the ring $(\mathcal{O}_{\mathbf{K}}^{2\delta})^\times$ of units. However, with this modification, Theorem 1.3 does not remain true, as an example at the end of section 3 will show.

As a special case, we want to consider fields of characteristic 0. Assume $\mathrm{char}K = 0$ and let p be the *characteristic exponent* of the residue field $\overline{K}$, i.e. $p = \mathrm{char}(\overline{K}) > 0$ or $p = 1$ if $\mathrm{char}(\overline{K}) = 0$. The *canonical decomposition* of the valuation v is defined as follows. Denote by $\Delta_{\mathbf{K}}$ the smallest convex subgroup of vK containing the value vp; note that the value set $\{n \cdot vp \mid n \in I\!N\}$ is cofinal in Δ. We write $\dot{v} := v_{\Delta_{\mathbf{K}}}$; this is called the *coarse valuation* assigned to v. Denote by $\dot{\mathbf{K}}$ the valued field $(K, \dot{v})$. The valuation ring $\mathcal{O}_{\dot{\mathbf{K}}}$ of $\dot{\mathbf{K}}$ is characterized as the smallest overring of $\mathcal{O}_{\mathbf{K}}$ in which p becomes a unit, i.e. $\mathcal{O}_{\dot{\mathbf{K}}}$ is the ring of fractions of $\mathcal{O}_{\mathbf{K}}$ with respect to the multiplicatively closed set $\{p^n \mid n \in I\!N\}$; consequently, the residue field $K\dot{v}$ is of characteristic 0. Note that $\dot{v} = v$ iff $p = 1$, and $\dot{v}$ is trivial iff $\Delta_{\mathbf{K}} = vK$. For $n \in I\!N$ we write $\mathbf{K}_n$ instead of $\mathbf{K}_{n\cdot vp}$.

Since $(K, \dot{v})$ has residue characteristic $\mathrm{char}K\dot{v} = 0$, every algebraic extension of a henselization of $(K, \dot{v})$ is tame. This is immediately seen from the second characterization of tame extensions given above; note that in this case, condition 3) is a consequence of the Lemma of Ostrowski. Hence, with $\Delta = \Delta_{\mathbf{K}}$, we obtain from Theorems 1.2 and 1.3 the following corollary:

Corollary 1.4 *Let* $\mathbf{K} = (K, v)$ *be a valued field of characteristic zero,* $\mathbf{L} = (L, v)$ *a henselian algebraic extension of* $\mathbf{K}$ *and* $\mathbf{F} = (F, v)$ *an arbitrary henselian extension of* $\mathbf{K}$*. The next statements are equivalent:*

i) $\mathbf{L}$ *is* $\mathbf{K}$*–embeddable into* $\mathbf{F}$*,*

ii) $\mathbf{L}_n$ *is* $\mathbf{K}_n$*–embeddable into* $\mathbf{F}_n$ *for each* $n \in I\!N$*.*

If also $F|K$ *is algebraic, then the next statements are equivalent:*

i) $\mathbf{L}$ *and* $\mathbf{F}$ *are* $\mathbf{K}$*-isomorphic.*

ii) $\mathbf{L}_n$ *and* $\mathbf{F}_n$ *are* $\mathbf{K}_n$*–isomorphic for each* $n \in I\!N$*.*

In section 4 below, we will give a special case of this corollary which includes the case of p–valued fields $\mathbf{K}$ and $\mathbf{L}$ with equal residue fields (Theorem 4.2). It only needs embeddings of the groups G^δ instead of embeddings of the full δ–structures and turns out to be a natural extension of Theorem 1.1. The other extreme, where only embeddings of the rings $\mathcal{O}_{\mathbf{K}}^\delta$ are needed, will be discussed in the following section; see Proposition 2.7 for an analogue of Theorem 1.2 in that case.

2 The case $\Delta = vK$

Let $\mathbf{K} = (K, v)$ be a valued field and $\mathbf{L} = (L, v_{\mathbf{L}})$ be a simple separable–algebraic extension of $\mathbf{K}$, i.e. $L = K(c)$ with $c \in L$ separable–algebraic over K. The monic minimal polynomial of c over K will be denoted by f. Let $\mathbf{F} = (F, v_{\mathbf{F}})$ be an arbitrary extension of $\mathbf{K}$ and denote by $\mathrm{Hom}_{\mathbf{K}}(\mathbf{L}, \mathbf{F})$ the set of $\mathbf{K}$–embeddings of $\mathbf{L}$ into $\mathbf{F}$ (this notation indicates valuation preserving embeddings). Now every K–embedding of L into F is simply given by sending c to some root z of f which is contained in F; consequently, there is a bijection between $\mathrm{Hom}_K(L, F)$ and $\{z \in F \mid f(z) = 0\}$. However, not every embedding $\sigma \in \mathrm{Hom}_K(L, F)$ is necessarily valuation preserving; if it is not, then $v_{\mathbf{F}} \circ \sigma \neq v_{\mathbf{L}}$, i.e. $v_{\mathbf{F}} \circ \sigma$ is a prolongation of v from K to L which is different from $v_{\mathbf{L}}$. We want to choose the element $c \in L$ so carefully that we may recognize from the value $v_{\mathbf{F}}\sigma c$ whether $v_{\mathbf{F}} \circ \sigma = v_{\mathbf{L}}$ or not. Since everything is trivial if v is a trivial valuation, we may always assume that K is not a finite field.

Lemma 2.1 *If $L|K$ is a simple separable–algebraic extension and K is not a finite field, then there are infinitely many primitive elements c generating L over K which satisfy $v_{\mathbf{L}}c = 0$ and $wc > 0$ for every prolongation w of v from K to L which is different from v. For such an element c, its monic minimal polynomial f belongs to $\mathcal{O}_{\mathbf{K}}[X]$ and all roots of f in an arbitrary valued field extension of $\mathbf{K}$ have value ≥ 0.*

Proof: Since $L|K$ is finite, there are only finitely many prolongations of v from K to L, and they are pairwise incomparable, i.e. the corresponding valuation rings are not comparable by inclusion. From the Chinese Remainder Theorem ([4], Proposition 1, p. 143) we may thus infer the existence of some element $d \in L$ which has residue 1 modulo $v_{\mathbf{L}}$ but residue 0 for all other prolongations of v from K to L.

Let c' be some primitive element generating L over K. Then by virtue of the theorem on primitive elements, for all but finitely many $\lambda \in K$, the element $d + \lambda c'$ will also be a generator for L over K. Since vK is cofinal in vL, there are infinitely many $\lambda \in K$ with $v\lambda > -wc'$ for every prolongation w of v to L. For such λ, the element $c = d + \lambda c'$ is a generator for L over K satisfying $v_{\mathbf{L}}c = 0$ and $wc > 0$ for every other prolongation w of v from K to L.

Since for every K–automorphism σ of L, the valuation $v_{\mathbf{L}} \circ \sigma$ is a prolongation of v from K to L, we see that in the algebraic closure of L equipped with an arbitrary prolongation of the valuation $v_{\mathbf{L}}$, all K–conjugates of c have value ≥ 0 which yields that also all coefficients of the minimal polynomial of c over K have value ≥ 0. $\square$

Now we choose c and f as in the preceding lemma. Then we have as an immediate consequence of the preceding lemma:

Lemma 2.2 *The map $\sigma \mapsto \sigma c$ is a bijection between $\mathrm{Hom}_{\mathbf{K}}(\mathbf{L}, \mathbf{F})$ and the set $\{z \in F \mid f(z) = 0 \wedge v_{\mathbf{F}}z = 0\}$.*

For large enough $\delta \in vK^{\geq 0}$ we will study the action of π_δ on the set $\{z \in F \mid f(z) = 0\}$; note that this set is a subset of $\mathcal{O}_{\mathbf{F}}$ by virtue of the last assertion of

Lemma 2.1. For $g \in \mathcal{O}_{\mathbf{K}}[X]$, let g^δ be the image of g in $\mathcal{O}_{\mathbf{K}}^\delta[X]$ obtained by an application of π_δ to the coefficients of g.

Let $n = \deg f$ and $a_1, \ldots, a_n$ be all roots of f in the algebraic closure $\tilde{F}$ of F; since f is separable and irreducible over K, all these roots are different. For every i, $1 \leq i \leq n$, let $\mu_i = \max\{\tilde{v}(a_i - a_j) \mid 1 \leq j \leq n \wedge j \neq i\}$ (where $\tilde{v}$ is a prolongation of $v_{\mathbf{F}}$ to $\tilde{F}$). Let $\delta_0 = v(\mathrm{discr} f)$ where $\mathrm{discr} f$ denotes the discriminant of f; note that $\delta_0 \in vK^{\geq 0}$ since $f \in \mathcal{O}_{\mathbf{K}}[X]$. Furthermore, observe that

$$\delta_0 = \sum_{i \neq j} \tilde{v}(a_i - a_j) \geq \sum_i \mu_i \geq \max_{i \neq j} \tilde{v}(a_i - a_j) \,. \tag{3}$$

The following lemma is a more precise version of the Hensel–Rychlik–Lemma; it holds for every normed, irreducible and separable polynomial $f \in \mathcal{O}_{\mathbf{K}}[X]$:

Lemma 2.3 *For every henselian extension field* $\mathbf{F}$ *of* $\mathbf{K}$ *and every* $\delta \in vK$, $\delta \geq \delta_0$, *the projection* π_δ *from* $\{z \in F \mid f(z) = 0\}$ *to* $\{x \in \mathcal{O}_{\mathbf{F}}^\delta \mid f^\delta(x) = 0\}$ *is injective and moreover, for every element* x *in the latter set, there is an element* z *in the former set such that* $z + \mathcal{M}_{\mathbf{F}}^{\delta-\delta_0} = x + \mathcal{M}_{\mathbf{F}}^{\delta-\delta_0}$. *In particular,* $v_{\mathbf{F}} z = 0$ *iff* x *is invertible.*

Proof: The projection is injective by virtue of (3). Now assume $x \in \mathcal{O}_{\mathbf{F}}^\delta$ such that $f^\delta(x) = 0$. Taking z' to be any element in F such that $\pi_\delta z' = x$, we obtain $v_{\mathbf{F}} f(z') > \delta$, hence in view of (3),

$$\sum_{1 \leq i \leq n} \tilde{v}(z' - a_i) = v_{\mathbf{F}} f(z') > \delta \geq \sum_{1 \leq i \leq n} \mu_i$$

showing that there must be at least one i_0 such that

$$\tilde{v}(z' - a_{i_0}) > \mu_{i_0} = \max\{\tilde{v}(a_{i_0} - a_j) \mid 1 \leq j \leq n \wedge j \neq i_0\} \,. \tag{4}$$

In fact, i_0 is the only such index; indeed, if $i_1 \neq i_0$ had the same property, then

$$\tilde{v}(a_{i_1} - a_{i_0}) > \min\{\mu_{i_0}, \mu_{i_1}\} \geq \tilde{v}(a_{i_1} - a_{i_0}) \,,$$

a contradiction.

The F–conjugates of a_{i_0} are all among the roots a_i. By virtue of Krasner's Lemma ([4], Corollaire 1, p. 190) and our hypothesis that $\mathbf{F}$ be henselian, from (4) we may conclude that $a_{i_0} \in F(z') = F$. Putting $z = a_{i_0}$, we compute

$$\tilde{v}(z' - z) = v_{\mathbf{F}} f(z') - \sum_{i \neq i_0} \tilde{v}(z' - a_i) > \delta - \sum_i \mu_i \geq \delta - \delta_0 \,;$$

this yields $z + \mathcal{M}_{\mathbf{F}}^{\delta-\delta_0} = z' + \mathcal{M}_{\mathbf{F}}^{\delta-\delta_0} = z' + \mathcal{M}_{\mathbf{F}}^{\delta} + \mathcal{M}_{\mathbf{F}}^{\delta-\delta_0} = x + \mathcal{M}_{\mathbf{F}}^{\delta-\delta_0}$. □

Corollary 2.4 *If* $\delta_1 \in vK^{\geq 0}$ *such that* $\delta_1 \geq \max_{i \neq j} \tilde{v}(a_i - a_j)$ *and* $\delta \geq \delta_1 + \delta_0$, *then the projection* π_{δ_1} *is a bijection between the sets* $\{z \in F \mid f(z) = 0\}$ *and* $\{x + \mathcal{M}^{\delta_1} \mid x \in \mathcal{O}_{\mathbf{F}}^\delta \wedge f^\delta(x) = 0\}$.

We have chosen the generating element c such that $v_{\mathbf{L}} c = 0$, hence for every $\delta \in \Delta^{\geq 0}$, the element $\pi_\delta c$ is invertible in $\mathcal{O}_{\mathbf{L}}^\delta$, and this property is maintained by every $\mathcal{O}_{\mathbf{K}}^\delta$–embedding of $\mathcal{O}_{\mathbf{L}}^\delta$ into $\mathcal{O}_{\mathbf{F}}^\delta$. In view of Lemmata 2.2 and 2.3, this proves

Lemma 2.5 *If $\mathcal{O}_{\mathbf{L}}^{\delta}$ is embeddable over $\mathcal{O}_{\mathbf{K}}^{\delta}$ into $\mathcal{O}_{\mathbf{F}}^{\delta}$ for large enough $\delta \in \Delta^{\geq 0}$, then $\mathbf{L}$ is $\mathbf{K}$-embeddable into $\mathbf{F}$.*

Now it remains to treat arbitrary separable–algebraic extensions $\mathbf{L}|\mathbf{K}$. The following compactness argument for algebraic extensions is well known:

Lemma 2.6 *Since $L|K$ is algebraic, we have $L = \bigcup_{i\in I} K_i$ where $K_i|K$ runs through all finite subextensions of $L|K$. If every K_i admits a K–embedding σ_i into F then there is a K–embedding σ of L into F and a subset $J \subset I$ with $L = \bigcup_{j\in J} K_j$ and $\forall j \in J : \sigma|_{K_j} = \sigma_j$. In particular, if all σ_i are valuation preserving, then so is σ.*

The basic idea of the proof is the fact that for every i, there are only finitely many K–embeddings of K_i into F, so at least one of them coincides with the restriction of σ_j for arbitrarily large overfields K_j of K_i.

By virtue of the preceding lemma, we may deduce the following embedding criterion from Lemma 2.5.

Proposition 2.7 *Let $\mathbf{K} = (K, v)$ be a valued field, $\mathbf{L} = (L, v)$ a separable–algebraic extension of $\mathbf{K}$ and $\mathbf{F} = (F, v)$ an arbitrary henselian extension of $\mathbf{K}$. Then the next statements are equivalent:*

i) $\mathbf{L}$ is $\mathbf{K}$-embeddable into $\mathbf{F}$.

ii) For each $\delta \in vK^{\geq 0}$, the ring $\mathcal{O}_{\mathbf{L}}^{\delta}$ is embeddable over $\mathcal{O}_{\mathbf{K}}^{\delta}$ into $\mathcal{O}_{\mathbf{F}}^{\delta}$.

If we take $\Delta = vK$ in Theorem 1.2 then its statement ii) implies the statement ii) of the above proposition. This shows that in the special case of $\Delta = vK$, the proposition proves Theorem 1.2.

3 Tame extensions

Let Δ be a convex subgroup of vK and assume in the sequel that $(L, v_\Delta)|(K, v_\Delta)$ is a finite tame extension of henselian fields. Then $Lv_\Delta|Kv_\Delta$ is a finite separable extension and thus simple, generated by some $c^\Delta \in Lv_\Delta$ with monic minimal polynomial f^Δ chosen in accordance with Lemma 2.1. Let $f \in K[X]$ be some monic polynomial with v_Δ–integral coefficients whose reduction modulo v_Δ is f^Δ (i.e. the coefficients of f^Δ are the v_Δ–residues of the corresponding coefficients of f). Using Hensel's Lemma, one may find a root $c \in L$ of f with v_Δ–residue c^Δ. By the fundamental inequality (2) we obtain $[K(c) : K] = [Kv_\Delta(c^\Delta) : Kv_\Delta]$ and

$$K(c)v_\Delta = Kv_\Delta(c^\Delta) = Lv_\Delta \quad \text{and} \quad v_\Delta K(c) = v_\Delta K \;.$$

Again by Hensel's Lemma, the K–embeddings of $K(c)$ into an arbitrary henselian field extension (F, v_Δ) of (K, v_Δ) are in 1–1 correspondence to the Kv_Δ–embeddings of $Kv_\Delta(c^\Delta)$ into Lv_Δ or in other words, reduction modulo v_Δ is a 1–1 correspondence between the roots of f and the roots of f^Δ. Note that corresponding roots have the same image under π_δ for every $\delta \in \Delta^{\geq 0}$. Furthermore, we observe that $\delta_0 := v(\mathrm{discr} f) \in \Delta^{\geq 0}$ since the v_Δ–residue of $\mathrm{discr} f$ is $\mathrm{discr} f^\Delta \neq 0$.

Since (K, v_Δ) is assumed to be henselian, every K–embedding of $K(c)$ into F will preserve the valuation v_Δ since by a non–preserving embedding, a second prolongation of v_Δ from K to $K(c)$ would be induced. Since v is the composition of v_Δ and $\overline{v}_\Delta$, a K–embedding of $K(c)$ into F preserves v if and only if the corresponding Kv_Δ–embedding of $Kv_\Delta(c^\Delta)$ into Fv_Δ preserves $\overline{v}_\Delta$.

At this point, we apply Lemma 2.5 to the extensions $(Kv_\Delta(c^\Delta), \overline{v}_\Delta)|(Kv_\Delta, \overline{v}_\Delta)$ and $(Fv_\Delta, \overline{v}_\Delta)|(Kv_\Delta, \overline{v}_\Delta)$. In view of our above consideration, we obtain the following lemma, where $\mathbf{K}(c)$ denotes the field $K(c)$ together with the restriction of the valuation $v_\mathbf{L}$.

Lemma 3.1 *If $\mathcal{O}^\delta_\mathbf{L}$ is embeddable over $\mathcal{O}^\delta_\mathbf{K}$ into $\mathcal{O}^\delta_\mathbf{F}$ for large enough $\delta \in \Delta^{\geq 0}$, then $\mathbf{K}(c)$ is $\mathbf{K}$-embeddable into $\mathbf{F}$.*

Consider the extension $L|K(c)$. We have noted already that $Lv_\Delta = K(c)v_\Delta$. If also $v_\Delta L = v_\Delta K(c) = v_\Delta K$, then by condition 3) on tame extensions it follows that $L = K(c)$. This enables us to prove:

Proposition 3.2 *Let $\mathbf{K} = (K, v)$ be a valued field, $\mathbf{L} = (L, v)$ a separable–algebraic extension of $\mathbf{K}$ and $\mathbf{F} = (F, v)$ an arbitrary henselian extension of $\mathbf{K}$. If Δ is a convex subgroup of vK such that (L, v_Δ) is a tame unramified extension of some henselization of (K, v_Δ), then the next statements are equivalent:*

i) $\mathbf{L}$ is $\mathbf{K}$-embeddable into $\mathbf{F}$.

ii) For each $\delta \in \Delta^{\geq 0}$, the ring $\mathcal{O}^\delta_\mathbf{L}$ is embeddable over $\mathcal{O}^\delta_\mathbf{K}$ into $\mathcal{O}^\delta_\mathbf{F}$.

Proof: We may assume without loss of generality that (K, v_Δ) is henselian. For, otherwise, consider a henselization (K', v_Δ) of (K, v_Δ) in (L, v_Δ). As (F, v) is henselian, (F, v_Δ) is henselian too, whence (K', v_Δ) may be identified with a subfield of (F, v_Δ). Since the residue field of both (K, v_Δ) and (K', v_Δ) is Kv_Δ, it follows that $\mathcal{O}^\delta_{(K',v)} = \mathcal{O}^\delta_\mathbf{K}$ for every $\delta \in \Delta$. Thus statement ii) does not change if we replace (K, v) by (K', v).

Now, with the additional assumption that (K, v_Δ) be henselian, our proposition follows from Lemma 3.1 by virtue of the compactness argument 2.6. □

Now we have to treat the case of $v_\Delta L \neq v_\Delta K$. Let $\alpha \in v_\Delta L \setminus v_\Delta K$ and assume that m is some natural number such that $m\alpha \in v_\Delta K$. If $a \in L$ with $v_\Delta a = \alpha$, then $v_\Delta a^m \in v_\Delta K$ and thus there is some $b \in K$ with $v_\Delta(a^m/b) = 0$. Then the v_Δ–residue $(a^m/b)/v_\Delta \in Lv_\Delta = K(c)v_\Delta$ is not zero, hence there is some $h \in K[X]$ with $(a^m b^{-1} h(c))/v_\Delta = 1$. By condition 2) on tame extensions, m is prime to p if $\mathrm{char}(Kv_\Delta) = p > 0$, hence in the henselian field (L, v_Δ), we may use Hensel's Lemma to deduce the existence of some element $a_0 \in L$ which satisfies $a_0^m = a^m b^{-1} h(c)$; putting $d = a_0/a \in L$ we get $bd^m = h(c)$. Note that we may choose h with v_Δ–integral coefficients since it only has to satisfy $h^\Delta(c^\Delta) = ((a^m b^{-1})/v_\Delta)^{-1} \in Kv_\Delta(c^\Delta)$ where h^Δ denotes the reduction of h modulo v_Δ. But then we may even assume $h \in \mathcal{O}_\mathbf{K}[X]$ since we may multiply both $h(c)$ and b with an element from K with large enough value in Δ; this will change neither the equation $bd^m = h(c)$ nor the values $v_\Delta b = m\alpha$ and $v_\Delta h(c) = 0$.

Since $L|K$ is a finite extension, the group $v_\Delta L/v_\Delta K$ is a finite torsion group, say
$$v_\Delta L/v_\Delta K = \mathbb{Z}\cdot(\alpha_1+v_\Delta K)\times\ldots\times\mathbb{Z}\cdot(\alpha_r+v_\Delta K) \tag{5}$$
where every α_i has finite order, say m_i. Using the above procedure, for $1\leq i\leq r$ we choose elements

- $d_i\in L$ with $b_i d_i^{m_i}=h_i(c)$, where
- $b_i\in K$ with $v_\Delta(b_i)=m_i\alpha_i$
- $h_i\in\mathcal{O}_{\mathbf{K}}[X]$ with $v_\Delta h_i(c)=0$.

Then $v_\Delta L = v_\Delta K(c,d_1,\ldots,d_r)$, and since $K(c)\subset K(c,d_1,\ldots,d_r)\subset L$ and $K(c)v_\Delta = Lv_\Delta$, we also have $Lv_\Delta = K(c,d_1,\ldots,d_r)v_\Delta$. From condition 2) on tame extensions it follows that $L=K(c,d_1,\ldots,d_r)$. On the other hand,
$$\begin{aligned}[L:K(c)] &\geq (v_\Delta L : v_\Delta K(c)) = (v_\Delta L : v_\Delta K) = m_1\cdot\ldots\cdot m_r\\ &\geq [K(c,d_1,\ldots,d_r):K(c)] = [L:K(c)]\;,\end{aligned}$$
whence $[L:K(c)]=m_1\cdot\ldots\cdot m_r$ which shows that the extensions
$$K(c,d_1,\ldots,d_{i-1},d_{i+1},\ldots,d_r)|K(c)\quad\text{and}\quad K(c,d_i)|K(c)$$
are linearly disjoint for every i, $1\leq i\leq r$.

Consequently, the K–embeddings of L into F are in 1–1 correspondence to the set of $r+1$–tuples $(z,t_1,\ldots,t_r)\in F^{r+1}$ satisfying $f(c)=0$ and $b_i t_i^{m_i}=h_i(z)$ for $1\leq i\leq r$. If such an embedding preserves the valuation v of $K(c)$, then it also preserves the valuation v of L; this is seen as follows. An embedding which preserves the valuation v of $K(c)$ will also preserve the valuation v_Δ of $K(c)$ and will moreover induce a valuation preserving Kv_Δ–embedding of $(Lv_\Delta,\overline{v}_\Delta)$ into $(Fv_\Delta,\overline{v}_\Delta)$ since $Lv_\Delta=K(c)v_\Delta$. On the other hand, $(K(c),v_\Delta)$ being an algebraic extension of the henselian field (K,v_Δ), it is also henselian which yields that every embedding of L into F which preserves the valuation v_Δ of $K(c)$ will also preserve the valuation v_Δ of L (since otherwise it induces a second prolongation of v_Δ from $K(c)$ to L). Hence indeed, an embedding which preserves the valuation v of $K(c)$ will also preserve the valuation $v=v_\Delta\circ\overline{v}_\Delta$ of L.

The next lemma summarizes what we have shown:

Lemma 3.3 *The map*
$$\sigma\mapsto(\sigma c,\sigma d_1,\ldots,\sigma d_r)$$
is a bijection between $\mathrm{Hom}_{\mathbf{K}}(\mathbf{L},\mathbf{F})$ *and the set*
$$\mathcal{Z}_{\mathbf{F}} := \{(z,t_1,\ldots,t_r)\in F^{r+1}\mid f(z)=0\wedge v_{\mathbf{F}}z=0\wedge\bigwedge_{1\leq i\leq r} b_i t_i^{m_i}=h_i(z)\}\;.$$

Now it remains to investigate the conditions for the existence of the elements $t_i\in F$. Firstly, we choose some $\delta_0\in\Delta^{\geq 0}$ as large as required in Lemma 3.1. Secondly, let us define $\mu=\max_{1\leq i\leq r} v_{\mathbf{L}}h_i(c)$. We know that $v_\Delta h_i(c)=0$; since $L|K$ is algebraic, this shows that μ lies in the convex hull of Δ in vL, so we may choose some $\delta_1\in\Delta$ with $\delta_1\geq\mu$. Thirdly, we define
$$\delta_2 := 2\cdot v[L:K(c)] = 2\cdot v(m_1\cdot\ldots\cdot m_r)\geq 0$$

and observe the following. If the characteristic of $\overline{K}$ is 0 or if it is $p > 0$ and prime to $m_1 \cdot \ldots \cdot m_r$, then $\delta_2 = 0$. If on the other hand, char$\overline{K} = p > 0$ and p is not prime to $m_1 \cdot \ldots \cdot m_r$, then by our condition that $(L, v_\Delta)|(K, v_\Delta)$ be a tame extension, it must follow that char$(Kv_\Delta) = 0$, whence $vp \in \Delta$ and consequently, $\delta_2 \in \Delta$. In either case, $\delta_2 \in \Delta$ and $\delta_2 \geq 2 \cdot v(m_i)$ for every i.

Lemma 3.4 *For every $\delta \in \Delta^{\geq 0}$, $\delta \geq \max(\delta_0, \delta_1 + \delta_2)$ and every henselian extension field $\mathbf{F}$ of $\mathbf{K}$, the set $\mathcal{Z}_{\mathbf{F}}$ is nonempty if the set*

$$\mathcal{Z}_{\mathbf{F}}^{\delta} := \{(x, y_1, \ldots, y_r) \in (\mathcal{O}_{\mathbf{F}}^{2\delta})^{\times} \times (G_{\mathbf{F}}^{\delta})^r \mid f^{\delta}(x) = 0 \wedge \bigwedge_{1 \leq i \leq r} b_i^{\delta} y_i^{m_i} = \Theta_\delta h_i^{2\delta}(x)\}$$

is nonempty.

Proof: Let $(x, y_1, \ldots, y_r) \in \mathcal{Z}_{\mathbf{F}}^{\delta}$. Choose $z' \in F$ such that $\pi_{2\delta} z' = x$. By Lemma 2.3 there is a root $z \in F$ of f such that $v_{\mathbf{F}} z = 0$ and

$$z + \mathcal{M}_{\mathbf{F}}^{\delta} = z' + \mathcal{M}_{\mathbf{F}}^{\delta} \,. \tag{6}$$

We want to show that for every triple $(b, m, h) = (b_i, m_i, h_i)$ with $1 \leq i \leq r$, there is some $t \in F$ such that $bt^m = h(z)$. Since $h \in \mathcal{O}_{\mathbf{K}}[X]$, equation (6) implies

$$h(z') + \mathcal{M}_{\mathbf{F}}^{\delta} = h(z) + \mathcal{M}_{\mathbf{F}}^{\delta} \,, \tag{7}$$

which by virtue of $v_{\mathbf{F}} h(z) \leq \delta$ implies $v_{\mathbf{F}} h(z') \leq \delta$ and thus,

$$\pi_\delta^* h(z') = \Theta_\delta \pi_{2\delta} h(z') = \Theta_\delta h^{2\delta}(x) \,.$$

Let $\tilde{y} \in F$ such that $\pi_\delta^* = y$. The equation $b^\delta y^m = \Theta_\delta h^{2\delta}(x) = \pi_\delta^* h(z')$ shows that $b\tilde{y}^m(1 + \mathcal{M}_{\mathbf{F}}^{\delta}) = h(z')(1 + \mathcal{M}_{\mathbf{F}}^{\delta})$, whence

$$b\tilde{y}^m(1 + \mathcal{M}_{\mathbf{F}}^{\delta_2}) = h(z')(1 + \mathcal{M}_{\mathbf{F}}^{\delta_2}) \tag{8}$$

in view of $\delta_2 \leq \delta$. On the other hand, equation (7) yields

$$h(z)(1 + \mathcal{M}_{\mathbf{F}}^{\delta_2}) = h(z')(1 + \mathcal{M}_{\mathbf{F}}^{\delta_2}) \tag{9}$$

by virtue of $\delta_2 \leq \delta - \delta_1 \leq \delta - v_{\mathbf{F}} h(z)$; note that the last inequality holds since $v_{\mathbf{F}} z = 0$ implies that $v_{\mathbf{F}} h_i(z) = v_{\mathbf{L}} h_i(c) \leq \delta_1$. From (8) and (9) we obtain $b\tilde{y}^m(1 + \mathcal{M}_{\mathbf{F}}^{\delta_2}) = h(z)(1 + \mathcal{M}_{\mathbf{F}}^{\delta_2})$ or equivalently,

$$v_{\mathbf{F}}(1 - h(z)(b\tilde{y}^m)^{-1}) > \delta_2 \geq 2vm \,.$$

By Newton's Lemma ([3], p. 22), we may refine 1 to some $\tilde{t} \in F$ satisfying $\tilde{t}^m = h(z)(b\tilde{y}^m)^{-1}$, and $t := \tilde{y}\tilde{t}$ satisfies $bt^m = h(z)$. □

From this lemma, we obtain Theorem 1.2 as follows.

Proof of Theorem 1.2.
The implication i) ⇒ ii) is trivial, so it remains to prove the implication ii) ⇒ i).

Let K' be a henselization of (K, v_Δ). We claim that $(K', v)_\delta = \mathbf{K}_\delta$ for every $\delta \in \Delta$. Indeed, for every $a \in \mathcal{O}_{(K',v)}$, there is some $a' \in K$ such that $v_\Delta(a - a') >$

$v_\Delta(a) \geq 0$; hence $v(a-a') > \Delta$ and $v(a/a'-1) > \Delta$ which shows that $\pi_\delta a = \pi_\delta a'$ and $\pi_\delta^* a = \pi_\delta^* a'$. As in the proof of Proposition 3.2, it now follows that we may assume without loss of generality (K, v_Δ) to be henselian.

By virtue of the compactness argument 2.6, we have to prove ii) $\Rightarrow$ i) only for the case of a finite extension $L|K$. By hypothesis, $(L, v_\Delta)|(K, v_\Delta)$ is a tame extension; hence we may assume that L is of the form that we have deduced above, and we may apply the preceding lemma. We know that for every $\delta \in \Delta^{\geq 0}$, the element $\pi_{2\delta} c \in (\mathcal{O}_{\mathbf{F}}^{2\delta})^\times$ satisfies $f^{2\delta}(\pi_{2\delta} c) = 0$ and for $\delta \geq \delta_1$, we have $b_i^\delta (\pi_\delta^* d_i)^{m_i} = \Theta_\delta h_i^{2\delta}(\pi_{2\delta} c)$. By a $\mathbf{K}_\delta$–embedding of the structure $\mathbf{L}_\delta$ into $\mathbf{F}_\delta$, the tuple $(\pi_{2\delta} c, \pi_\delta^* d_1, \ldots, \pi_\delta^* d_r)$ is thus mapped to an element of the set $\mathcal{Z}_{\mathbf{F}}^\delta$. Now Lemma 3.4 together with Lemma 3.3 proves the existence of a $\mathbf{K}$–embedding of $\mathbf{L}$ into $\mathbf{F}$. $\square$

Example.
We will construct henselian fields $\mathbf{L}$ and $\mathbf{F}$ of characteristic 0 and residue characteristic $p > 2$, both finite extensions of a common henselian subfield $\mathbf{K}$ but not isomorphic over $\mathbf{K}$, such that for every $\delta = n \cdot vp$, $n \in \mathbb{N}$, the structures $(\mathcal{O}_{\mathbf{L}}^{2\delta}, G_{\mathbf{L}}^\delta, \Theta_\delta^\times)$ and $(\mathcal{O}_{\mathbf{F}}^{2\delta}, G_{\mathbf{F}}^\delta, \Theta_\delta^\times)$ are isomorphic over $(\mathcal{O}_{\mathbf{K}}^{2\delta}, G_{\mathbf{K}}^\delta, \Theta_\delta^\times)$, where $\Theta_\delta^\times$ denotes the restriction of Θ_δ to the units of the ring $\mathcal{O}^{2\delta}$.

Let $p > 2$ be any prime number and v be the p–adic valuation on the field $\mathbb{Q}_p$ of p–adic numbers. Take $\varepsilon \neq 1$ to be a p–th root of unity. Then $\mathbb{Q}_p(\varepsilon)$ has residue field $\mathbb{F}_p$ like $\mathbb{Q}_p$ and contains an element π such that $\pi^{p-1} = p$ and that $v\pi$ is the smallest positive element in $v\mathbb{Q}_p(\varepsilon)$. Assume x to be transcendental over $\mathbb{Q}$. Let $\mathbf{K}$ be a henselization of the valued field $(\mathbb{Q}(\varepsilon, x), v)$ where $vx \gg v\mathbb{Q}_p(\varepsilon)$. The residue field $\overline{K}$ is still equal to $\mathbb{F}_p$, and the value group vK is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, ordered lexicographically.

Let a be a root of the polynomial $f(X) = X^p + X - \frac{1}{p^2}$. Then $va = -\frac{2}{p} vp \notin vK$ and consequently, the extension $\mathbf{K}(a)|\mathbf{K}$ is purely ramified; the residue field $\overline{K(a)}$ is still equal to $\mathbb{F}_p$. Moreover, a is the only root of f in $K(a)$ (hence $K(a)|K$ is not cyclic). This is seen as follows. Assume that $b \neq a$ is another root of f. Then the element $z = \frac{b-a}{a\pi}$ satisfies the equation

$$z^{p-1} + \sum_{i=1}^{p-2} \binom{p}{i} \frac{\pi^i}{p} z^i + 1 + \frac{1}{pa^{p-1}} = 0 \,. \tag{10}$$

All coefficients in the sum are elements of the valuation ideal since $v\pi > 0$ and every binomial coefficient $\binom{p}{i}$ is divisible by p for $1 \leq i < p$. Moreover, $vpa^{p-1} = vp + (p-1)va = (1 - \frac{2(p-1)}{p})vp = \frac{2-p}{p} vp < 0$ since $p > 2$. Consequently, the reduction of equation (12) modulo v gives $\overline{z}^{p-1} = -1$ which is not possible for $\overline{z} \in \mathbb{F}_p$ since $p > 2$. This shows that $b \notin K(a)$.

Now choose elements $d_1, d_2 \in \tilde{K}$ such that $d_1^p = ax$ and $d_2^p = \varepsilon ax$. Both elements have value $\frac{1}{p} v(ax) \notin vK(a)$. Thus again, $\mathbf{L} := \mathbf{K}(a, d_1) = \mathbf{K}(d_1)$ and $\mathbf{F} := \mathbf{K}(a, d_2) = \mathbf{K}(d_2)$ are purely ramified extensions of $\mathbf{K}(a)$, and their residue fields are still equal to $\overline{K} = \mathbb{F}_p$. As before, a is the only root of f in $\mathbf{L}$ and $\mathbf{F}$.

Assume that there is an isomorphism σ from L onto F over K. Then $\sigma a = a$. Consequently, the equation

$$\left(\frac{d_2}{\sigma d_1}\right)^p = \frac{\varepsilon a x}{(\sigma a) x} = \varepsilon$$

shows that ε has a p–th root η in $K(d_2)$. But η is a primitive p^2–th root of unity, and $\eta \notin \mathbb{Q}_p(\varepsilon)$. Hence $\eta \notin K$. Since $K(\eta)|K$ is a cyclic extension of degree p, we also find $\eta \notin K(a)$. This yields $K(a,d_2) = K(a,\eta)$, but $\frac{1}{p}v(ax) \notin vK(a,\eta)$, a contradiction. Consequently, we have shown that L and F are not isomorphic over K.

Now let us consider the mixed δ–structures. First we observe that $\mathcal{O}^{2\delta}_{\mathbf{L}} = \mathcal{O}^{2\delta}_{\mathbf{K}(a)} = \mathcal{O}^{2\delta}_{\mathbf{F}}$ since the residue fields of $\mathbf{L}$ and $\mathbf{F}$ modulo $v_{\Delta_{\mathbf{K}}}$ both are henselizations of $\mathbf{K}(a)$. Thus, it remains to show that $G^{\delta}_{\mathbf{L}}$ and $G^{\delta}_{\mathbf{F}}$ are isomorphic over the group compositum

$$H := G^{\delta}_{\mathbf{K}} \,.\, \Theta^{\times}_{\delta}(\mathcal{O}^{2\delta}_{\mathbf{L}})^{\times} \;.$$

For every element $a \in L$ there exists some $\nu \in \{0,\ldots,p^2-1\}$ and $c \in K$ such that $ad_1^{\nu}c$ is a 1–unit, i.e. an element of $1+\mathcal{M}_L$. Since $\pi^*_\delta(1+\mathcal{M}_L) \subset \Theta^{\times}_{\delta}(\mathcal{O}^{2\delta}_{\mathbf{L}})^{\times}$, this shows that $(\pi^*_\delta a)H = (\pi^*_\delta d_1^{\nu})H$. A similar argument works for $\mathbf{F}$ in the place of $\mathbf{L}$. Hence the quotient groups $G^{\delta}_{\mathbf{L}}/H$ and $G^{\delta}_{\mathbf{F}}/H$ are both cyclic of order p^2 and are generated by $\pi^*_\delta d_1 H$ resp. $\pi^*_\delta d_2 H$. Since

$$(\pi^*_\delta d_1)^{p^2} = \pi^*_\delta(ax)^p = \pi^*_\delta(\varepsilon ax)^p = (\pi^*_\delta d_2)^{p^2} \;,$$

the groups $G^{\delta}_{\mathbf{L}}$ and $G^{\delta}_{\mathbf{F}}$ are indeed isomorphic over H. However, for $\delta \geq 2vp$ they are not isomorphic over the group compositum

$$H' := G^{\delta}_{\mathbf{K}} \,.\, \Theta_{\delta}\{x \in \mathcal{O}^{2\delta}_{\mathbf{K}} \mid x^2 \neq 0\} \;.$$

Indeed, for $\delta \geq 2vp$ the group H' contains $\pi^*_\delta a = \Theta_\delta \pi_{2\delta} a$ and $\pi^*_\delta \varepsilon a = \Theta_\delta \pi_{2\delta}\varepsilon a$. Thus $G^{\delta}_{\mathbf{L}}/H'$ and $G^{\delta}_{\mathbf{F}}/H'$ are both of order p, and every isomorphism τ over H' would give

$$\left(\frac{\pi^*_\delta d_2}{\tau\pi^*_\delta d_1}\right)^p = \frac{\pi^*_\delta \varepsilon ax}{\pi^*_\delta ax} = \pi^*_\delta \varepsilon \;.$$

This would indicate the existence of some element $\tilde\eta \in K(d_2)$ which satisfies $v(\tilde\eta^p - \varepsilon) > \delta \geq 2vp$ and $v\tilde\eta = v\varepsilon = 0$. By virtue of Newton's Lemma, this would yield the existence of a p–th root of ε in $K(d_2)$, a contradiction.

4 The special case of radical extensions

Let $(L,v_\Delta)|(K,v_\Delta)$ be as in the preceding section, but assume in addition that $Lv_\Delta = Kv_\Delta$. Then the element c may be omitted. For the value groups, assume again the equation (5). For the choice of d_i, b_i, h_i observe that $(a_i^{m_i}/b_i)/v_\Delta \in Lv_\Delta = Kv_\Delta$, so after multiplication of b_i with a suitable element from K we may assume that the v_Δ–residue of $a_i^{m_i}/b_i$ is 1, hence we may choose $h_i = 1$ and we have $b_i d_i^{m_i} = 1$.

Another interesting case is given when $Lv_\Delta|Kv_\Delta$ is a nontrivial extension, but (L,v) is still generated over some henselization of (K,v) by radicals d_i with $d_i^{m_i} \in K$. Note that this condition is satisfied already if $(Lv_\Delta,\overline{v}_\Delta)$ is generated over some henselization of $(Kv_\Delta,\overline{v}_\Delta)$ by radicals over Kv_Δ. Indeed, by our general assumption that (L,v_Δ) be a tame extension of some henselization of (K,v_Δ), we know that $Lv_\Delta|Kv_\Delta$ is separable, so by Hensel's Lemma, radicals generating this residue field extension may be lifted up to radicals over K in L.

For both cases, we obtain the following lemma from the proof given in the last section. Since now δ has only to run through the multiples of the values vm_i, we may restrict δ to a (not necessarily convex) subgroup Δ_p defined in the following way, where $p = \mathrm{char}\overline{K}$:

$$\Delta_p := \begin{cases} \{n \cdot vp \mid n \in I\!N\} & \text{if } \mathrm{char}K = 0 \text{ and } vp \in \Delta \\ \{0\} & \text{otherwise.} \end{cases} \tag{11}$$

Lemma 4.1 *Let Δ be a convex subgroup of vK such that $(L, v_\Delta)|(K, v_\Delta)$ is a finite tame extension of henselian fields. Assume that $Lv_\Delta = Kv_\Delta$ or that (K, v) is henselian and $L|K$ is generated by radicals. Then $\mathbf{L}$ is $\mathbf{K}$–embeddable into $\mathbf{F}$ if $G^\delta_{\mathbf{L}}$ is $G^\delta_{\mathbf{K}}$–embeddable into $G^\delta_{\mathbf{F}}$ for some $\delta \in \Delta_p$ satisfying $\delta \geq 2 \cdot v[L : K]$.*

From this lemma (and the corresponding special case of Lemma 3.3), we obtain the following special case of Theorem 1.2 and Theorem 1.3:

Theorem 4.2 *Let $\mathbf{K} = (K, v)$ be a valued field, $\mathbf{L} = (L, v)$ a henselian algebraic extension of $\mathbf{K}$ and $\mathbf{F} = (F, v)$ an arbitrary henselian extension of $\mathbf{K}$. If Δ is a convex subgroup of vK such that (L, v_Δ) is a tame extension of some henselization of (K, v_Δ) satisfying in addition that $Lv_\Delta = Kv_\Delta$ or that (L, v) is generated over some henselization of (K, v) by radicals d_i with $d_i^{m_i} \in K$, then the next statements are equivalent:*

i) $\mathbf{L}$ *is* $\mathbf{K}$*–embeddable into* $\mathbf{F}$*,*

ii) $G^\delta_{\mathbf{L}}$ *is* $G^\delta_{\mathbf{K}}$*–embeddable into* $G^\delta_{\mathbf{F}}$ *for each* $\delta \in \Delta_p$*,*

iii) $K \cap L^n \subset K \cap F^n$ *for each* $n \in I\!N$*.*

If also $\mathbf{F}$ *is a henselian algebraic extension of* $\mathbf{K}$ *satisfying the same conditions as* $\mathbf{L}$*, then the next statements are equivalent:*

i) $\mathbf{L}$ *and* $\mathbf{F}$ *are* $\mathbf{K}$*–isomorphic,*

ii) $G^\delta_{\mathbf{L}}$ *and* $G^\delta_{\mathbf{F}}$ *are* $G^\delta_{\mathbf{K}}$*–isomorphic for each* $\delta \in \Delta_p$*,*

iii) $K \cap L^n = K \cap F^n$ *for each* $n \in I\!N$*.*

Note that in many cases, the assumption $d_i^{m_i} \in K$ may be replaced by the assumption that L be generated by radicals over some henselization K^H of K, i.e. $d_i^{m_i} \in K^H$:

Lemma 4.3 *Assume that d^m is an element of some henselization K^H of (K, v). If $\mathrm{char}\overline{K} = p > 0$ and $p|m$, then assume in addition that $\mathrm{char}K = 0$ and that $(Kv_\Gamma, \overline{v}_\Gamma)$ is henselian, where Γ denotes the largest convex subgroup of vK not containing vp. Then there is some $d_0 \in K^H$ with $d_0^m \in K$ and $K^H(d_0) = K^H(d)$.*

The proof is left to the reader. Note that in the case of $\mathrm{char}\overline{K} = p > 0$ and $p|m$, our additional assumption yields that K^H may be viewed as a henselization of (K, v_Γ) and consequently, for every $n \in I\!N$, there is some $b_n \in K$ with $v(d^m - b_n) \geq n \cdot vp$; choosing n such that $n \cdot vp > 2 \cdot vm$, one may thus use Newton's Lemma to find d_0.

Theorem 4.2 extends Theorem 1.1; to prove this we have to show that the hypotheses of the former are a consequence of those of the latter theorem. We will do this for a more general class of valued fields. A valued field (K, v) is called *finitely ramified* if vK has a smallest positive element $v\pi$ and there is a prime p such that vp is a finite multiple of $v\pi$; then $\mathrm{char}K = 0$ and $\mathrm{char}\overline{K} = p$. Note that finitely ramified fields satisfy the hypothesis of the preceding lemma since $\Gamma = \{0\}$ and thus, $(Kv_\Gamma, \overline{v}_\Gamma)$ is a trivially valued field which is always henselian.

Every p–valued field is finitely ramified. Hence the following lemma links Theorem 1.1 to Theorem 4.2:

Lemma 4.4 *Let $(L, v)|(K, v)$ be an algebraic extension of finitely ramified fields such that vL and vK have the same smallest positive element and $\overline{L} = \overline{K}$. If (L, v) is henselian, then L is generated over some henselization of (K, v) by radicals over K.*

The proof is left to the reader; the main ingredients are Newton's Lemma and the fact that finitely ramified fields are defectless, i.e. every finite extension of henselian finitely ramified fields satisfies equation (1).

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Correction to An Isomorphism Theorem for Henselian Algebraic Extensions of Valued Fields

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In the example on page 123 of the article [1], the correct equation for π should be $\pi^{p-1} = -p$. As a consequence of the forgotten minus sign, the succeeding choice of the polynomial f and the corresponding verification is false. The intention was to provide a non-cyclic extension of $\mathbb{Q}_p(\varepsilon)$ of degree p, where $\varepsilon \neq 1$ denotes a p-th root of unity. The polynomial given in our example may be replaced by the polynomial $X^p + \pi X + \pi$. Let a be a root of this polynomial and let $\mathbf{K} = (\mathbb{Q}_p(\varepsilon, x), v)$ as in the example. Then $va = \frac{1}{p} v\pi \notin vK$ and consequently, the extension $\mathbf{K}(a)|\mathbf{K}$ is purely ramified; the residue field $\overline{K(a)}$ is equal to $\mathbb{F}_p$. Moreover, a is the only root of f in $K(a)$ (hence $K(a)|K$ is not cyclic). Indeed, assume that $b \neq a$ is another root of f. Then the element $z = \frac{b-a}{a}$ satisfies the equation

$$z^{p-1} + \sum_{i=0}^{p-2} \binom{p}{i+1} z^i + \frac{\pi}{a^{p-1}} = 0 \,. \tag{12}$$

All binomial coefficients $\binom{p}{i+1}$ are divisible by p for $0 \leq i \leq p-2$. Moreover, $v\pi a^{-(p-1)} = pva - (p-1)va = va < vp$. Consequently, $vz = \frac{1}{p-1} va$ which is not an element of $vK(a)$ since $p > 2$. This shows that $b \notin K(a)$.

We are very endebted to F. Lorenz for giving us the idea how to find the right polynomial. His argument was as follows. Given a polynomial f, its discriminant is by definition a square in its splitting field. So we have to find an irreducible polynomial of degree p over $\mathbb{Q}_p(\varepsilon)$ whose discriminant is not a square in $\mathbb{Q}_p(\varepsilon)$. Indeed, since the extension by one of its roots a is of odd degree p, this discriminant will still not be a square in $\mathbb{Q}_p(\varepsilon, a)$, showing that $\mathbb{Q}_p(\varepsilon, a)$ is not the splitting field of f. This yields that the extension $\mathbb{Q}_p(\varepsilon, a)|\mathbb{Q}_p(\varepsilon)$ is not cyclic.

For polynomials of the form $X^n + cX + d$, the discriminant may be computed by the formula

$$\mathrm{discr}\, f = (-1)^{n(n-1)/2}(n^n d^{n-1} + (1-n)^{n-1}c^n)$$

(cf. Lorenz [2], p. 316). With this formula, one finds that the discriminant of $X^p + \pi X + \pi$ is equal to $\pm(p^p \pi^{p-1} + (1-p)^{p-1}\pi^p)$. So $v\,\mathrm{discr}\, f = pv\pi$ which is not divisible by 2 in $v\mathbb{Q}_p(\varepsilon)$ and vK since $v\pi$ is the smallest element in both value groups.

Later in the same example, there is a paragraph which should better read as follows: "For every element $b \in L$ there exists some $\nu \in \{0, \ldots, p^2-1\}$ and $c \in K$ such that $b^{-1}d_1^\nu c$ is a 1–unit, i.e. an element of $1 + \mathcal{M}_L$. Since $\pi_\delta^*(1 + \mathcal{M}_L) \subset \Theta_\delta^\times(\mathcal{O}_{\mathbf{L}}^{2\delta})^\times$, this shows that $(\pi_\delta^* b)H = (\pi_\delta^* d_1^\nu)H$."

Finally, the compactness argument in Lemma 2.6 should be formulated in a slightly different way: "Since $L|K$ is algebraic, we have $L = \bigcup_{i\in I} K_i$ where $K_i|K$ runs through all finite subextensions of $L|K$. If every K_i admits a K–embedding σ_i in F then there is a K–embedding σ of L in F such that

$$\forall i \in I: \quad \sigma|_{K_i} = \sigma_j|_{K_i} \text{ for some } j \in I \text{ such that } K_i \subset K_j\ . \tag{13}$$

In particular, if all σ_i are valuation preserving, then so is σ."

References

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