A general framework for fixed point theorems, and more

Franz-Viktor Kuhlmann
joint work with Katarzyna Kuhlmann

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This talk is dedicated to the memory of **Serban Basarab**, a collaborator and friend, who passed away all of a sudden on July 14, 2014.

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For most FPTs a notion of “completeness” is needed.

Banach's FPT for metric spaces

On a metric space (X, d), a function $f : X \to X$ is called strictly contracting if there is a positive real number $0 < c < 1$ such that $d(fx, fy) \le cd(x, y)$ for all $x, y \in X$.

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Theorem (Banach FPT)

Every strictly contracting function on a complete metric space (X, d) has a unique fixed point.

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For $x,y \in X$,

$$B(x,y) \;:=\; B_{u(x,y)}(x) \;=\; B_{u(x,y)}(y)$$

is the smallest ball containing x and y.

Spherically complete ultrametric spaces

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An ultrametric space (X, u) is spherically complete if the intersection of every nest of balls is nonempty.

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Theorem (S. Prieß-Crampe)

Take a function f on a spherically complete ultrametric space (X,u) which is contracting, that is,

$$u(fx,fy) < u(x,y)\,.$$

Then f admits a unique fixed point.

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Theorem (S. Prieß-Crampe)

Take a function f on a spherically complete ultrametric space (X, u) such that:

(1) $u(fx, fy) \leqslant u(x, y)$ (f *is nonexpanding*),

(2) $u(fx, f^2x) < u(x, fx)$ (f *is contracting on orbits*).

Then f admits a fixed point.

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Condition (AT1) says that the approximation fy of z' from within the image of f can be improved, and condition (AT2) says that this can be done in a somewhat "continuous" way.

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Assume that $z' \in Y'$ is an attractor for f and that (Y, u) is spherically complete. Then $z' \in f(Y)$.

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Assume that $z' \in Y'$ *is an attractor for* f *and that* (Y, u) *is spherically complete. Then* $z' \in f(Y)$.

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Theorem (FVK)

Assume that f *is immediate and that* (Y, u) *is spherically complete. Then* f *is surjective and* (Y', u') *is spherically complete. Moreover, for every* $y \in Y$ *and every ball* B' *in* Y' *containing* fy*, there is a ball* B *in* Y *containing* y *and such that* $f(B) = B'$.

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- In 2011, M. Kostanek and P. Waszkiewicz unified Banach, Caristi, Knaster-Tarski and Bourbaki-Witt FPTs in their paper *Reconciliation of elementary order and metric fixpoint theorems*.

Origin of our work

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Katarzyna has studied spaces of $\mathbb{R}$-places together with I. Efrat, M. Marshall, T. Banakh, M. Machura and FVK.

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Structures which look like fractals, but cannot be described by *the usual* definitions for fractals, were also found in:

- algebraic geometry;
- complex spaces;
- modal logic.

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Can we come up with fixed point theorems that could help us with defining generalized notions of fractals?

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Take a spherically complete ball space $(X, \mathcal{B})$ and a function $f : X \to X$.

- *If for every ball $B \in \mathcal{B}$, $f(B)$ contains an f-contracting ball, then f has a fixed point in every ball.*

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- *If $X \in \mathcal{B}$ and for every ball $B \in \mathcal{B}$, $f(B)$ is an f-contracting ball, then f has a* **unique** *fixed point.*

Adjusting the ball space to the given function

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$$\mathcal{B}^f := \{B \in \mathcal{B} \mid B \text{ is } f\text{-closed}\} .$$

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As we will see later, some important ball spaces $(X, \mathcal{B})$ have the property that the intersection over a nest of balls is again a ball. It then follows that the same is true for $(X, \mathcal{B}^f)$, provided that $\mathcal{B}^f$ is nonempty.

FPT for spherically complete ball spaces, II

Theorem (General FPT)

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- But then, $\mathcal{N} \cup \{B\}$ is a larger nest, contradicting the maximality of $\mathcal{N}$.
- Hence there is a fixed point in $\bigcap \mathcal{N} \subseteq B_0$.

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- By Zorn's Lemma, there is a minimal ball B_0 in $\mathcal{B}'$.
- By condition (2), B_0 must contain a fixed point of f.

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If $x \neq fx$, then

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Application to metric spaces

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For a subset $S \subseteq \mathbb{R}^+$ define:

$$\mathcal{B}_S = \{B_r(x) : x \in X, r \in S\}.$$

Theorem

Take a set $S \subseteq \mathbb{R}^+$ which has 0 as its unique limit point. A metric space (X, d) is complete if and only if the ball space $(X, \mathcal{B}_S)$ is spherically complete.

Applications to metric spaces

Theorem (Banach FPT)

Every strictly contracting function on a complete metric space (X, d) has a unique fixed point.

Choose $x \in X$ and set $d := d(x, fx)$. Take $k \in \mathbb{N}$ such that $c^k < \frac{1}{2(1-c)}$. Then the balls

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are f-closed, and

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Fact

The only field which is complete with respect to the metric d is $\mathbb{R}$. Since $\mathbb{R}$ is cut complete, that is, for every cut (C, D) in $(\mathbb{R}, <)$, C has a largest or D has a smallest element, it is a spherically complete ball space, where the balls are the nonempty closed bounded intervals.

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- Then $\bigcap \mathcal{N} \neq \emptyset$ if and only if there is $x \in X$ such that $a_\nu \leqslant x \leqslant b_\nu$ for every ν.
- If $\bigcap \mathcal{N} \neq \emptyset$, then $\mathcal{N}$ determines a cut C which is not filled in X.

Asymmetric cuts

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An ordered set in which every cut is asymmetric is called symmetrically complete.

For an ordered group or field X we take $\mathcal{B}$ to be the collection of all nonempty closed bounded intervals of X. Then the following theorem holds:

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Theorem

$(X, \mathcal{B})$ is spherically complete if and only if it is symmetrically complete.

A function $f : X \to X$ on an ordered group X is nonexpanding if $|fx - fy| \leq |x - y|$ for all $x, y \in X$.

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The function f is strictly contracting on orbits if there is a positive rational number $\frac{m}{n} < 1$ with $m, n \in \mathbb{N}$ such that $n|fx - f^2x| \leq m|x - fx|$ for all $x \in X$.

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Theorem

Take a symmetrically complete ordered group X. Then every nonexpanding function on X which is strictly contracting on orbits has a fixed point.

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- (2013) In joint work with S. Shelah we extended his result to ordered abelian groups and characterized all symmetrically complete ordered abelian groups and fields. (To appear in Israel J. Math.)

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A collection of sets is a centered system if the intersection of any finite subcollection is nonempty.

If X is a topological space, then we will consider its associated ball space $(X, \mathcal{B})$ where $\mathcal{B}$ consists of all nonempty closed sets.

Directed systems

We will also need directed systems when we deal with partially ordered sets (posets). A directed system in a poset is a nonempty subset in which every two elements have an upper bound which is also contained in the subset.

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A poset $(X, \leqslant)$ is called a directed complete partial order (DCPO) if every directed system of elements has a supremum. This condition implies that the poset is chain complete, that is, every nonempty chain of elements of X has a supremum in X. The two properties are equivalent if the axiom of choice is assumed (which we always do, as we are also working with Zorn's Lemma).

Directed systems of balls

In ball spaces we are concerned with intersections of balls, so we define a directed system of balls to be a nonempty set of balls such that for every two balls in this set there is a ball in the set that is contained in their intersection.

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Moreover, observe that in a topological space arbitrary intersections of closed sets are again closed. What about ball spaces in which all (nonempty) intersections of nests, directed systems, or centered systems, are again balls?

Hierarchy of spherical completeness

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We will also write $\mathbf{S}^*$ for $\mathbf{S}_4^c$ because this turns out to be the "star" (the strongest) among the ball spaces:

Hierarchy of spherical completeness

$$
\begin{array}{ccccccc}
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\Uparrow & & \Uparrow & & \Uparrow & & \\
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\mathbf{S}_4 & \Leftarrow & \mathbf{S}_4^d & \Leftarrow & \mathbf{S}_4^c & = & \mathbf{S}^*
\end{array}
$$

Spherical completeness

spaces	balls	completeness property	equiv. to
ultrametric spaces	all ultrametric balls	spherically complete	$\mathbf{S}_1$
metric spaces	metric balls with radii in suitable sets of positive real numbers	complete	$\mathbf{S}_1$
ordered abelian groups and fields	all intervals $[a, b]$ with $a \leq b$	symmetrically complete	$\mathbf{S}_1$
topological spaces	all nonempty closed sets	compact	$\mathbf{S}_1^c$
posets	final segments $\uparrow a = \{b \mid a \leq b\}$	directed complete	$\mathbf{S}_4^d$
lattices	final segments $\uparrow a$, initial segments $\downarrow a = \{b \mid a \geq b\}$, and intervals $[a, b]$, $a \leq b$	complete	$\mathbf{S}^*$

Connection with posets

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Proposition

The ball space $(X, \mathcal{B})$ is $\mathbf{S}_3$ if and only if $(\mathcal{B}, <)$ is chain complete, and it is $\mathbf{S}_3^d$ if and only if $(\mathcal{B}, <)$ is directed complete.

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As we always assume the axiom of choice, we have:

Proposition

A poset is directed complete if and only if it is chain complete.

Corollary

A ball space is S_3 if and only if it is S_3^d.

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1) If the ball space $(X, \mathcal{B})$ is finitely intersection closed, then $\mathbf{S}_i^d$ is equivalent to $\mathbf{S}_i^c$, for $i = 1, 2, 3, 4$.

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Theorem

1) If the ball space $(X, \mathcal{B})$ is finitely intersection closed, then $\mathbf{S}_i^d$ is equivalent to $\mathbf{S}_i^c$, for $i = 1, 2, 3, 4$.
2) If the ball space $(X, \mathcal{B})$ is intersection closed, then all properties in the hierarchy are equivalent.

- The ball space associated with a topological space is intersection closed.

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A complete lattice is a poset $(X, \leqslant)$ in which every subset has infimum and supremum.

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Proposition

The associated ball space of the reals is $\mathbf{S}^*$. For all other densely ordered abelian groups or fields the associated ball space can at best be $\mathbf{S}_1$ or $\mathbf{S}_2$.

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Take a topological space X. The following are equivalent:
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Take a topological space X. The following are equivalent:
a) X is compact,
b) the ball space associated with X is $\mathbf{S}_1$,
c) the ball space associated with X is $\mathbf{S}^*$.

Theorem

Take a compact space X and a function $f : X \to X$. If for every closed and f-closed set B with at least two elements there is a nonempty closed and f-closed set $B' \subsetneq B$, then f has a fixed point in every closed and f-closed set.

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For a nest $\mathcal{N}$ in $\mathcal{B}$, the set $\bigcap \mathcal{N}$ is closed and $f(\bigcap \mathcal{N}) \subseteq \mathcal{N}$, so $\bigcap \mathcal{N} \in \mathcal{B}$.

Applications to topological spaces

We consider a family

$$\mathcal{B} = \{B \subseteq X \mid B \text{ is closed and } f\text{-closed}\}.$$

For a nest $\mathcal{N}$ in $\mathcal{B}$, the set $\bigcap \mathcal{N}$ is closed and $f(\bigcap \mathcal{N}) \subseteq \mathcal{N}$, so $\bigcap \mathcal{N} \in \mathcal{B}$.

Theorem (General FPT)

Assume that there is a ball space structure $(X, \mathcal{B})$ on X for which the following conditions are satisfied:
(1) *every $B \in \mathcal{B}$ is f-closed,*
(2) *every nonsingleton $B \in \mathcal{B}$ properly contains some $B' \in \mathcal{B}$,*
(3) *the intersection of every nest of balls in $\mathcal{B}$ is a singleton or contains some $B \in \mathcal{B}$.*
Then f admits a fixed point in every ball.

Applications to topological spaces

Theorem

Take a compact space X and a closed function $f : X \to X$. If every nonempty closed and f-closed set B in X is f-contracting, then f has a **unique** *fixed point in X.*

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Theorem

Take a spherically complete ball space $(X, \mathcal{B}^f)$ and a function $f : X \to X$. If $X \in \mathcal{B}^f$ and for every ball $B \in \mathcal{B}^f$, $f(B)$ is f-contracting, then f has a **unique** *fixed point.*

Applications to topological spaces

Take a continuous function $f : X \to X$ on a topological space X. An open cover $\mathcal{U}$ of X is called J-contractive for f if for every $U \in \mathcal{U}$ there is $U' \in \mathcal{U}$ such that $f(clU) \subseteq U'$.

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A continuous function $f : X \to X$ on a topological space X is called J-contraction if any open cover $\mathcal{U}$ has a finite open J-contractive refinement for f.

Proposition (J. Steprans, S. Watson, W. Just)

Let f be J-contraction on a connected compact Hausdorff space.
1) If B is a closed and f-closed subset of X, then the restriction of f to B is also a J-contraction.

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Theorem (J. Steprans, S. Watson, W. Just)

If f is a J-contraction on any connected compact Hausdorff space, then f has a unique fixed point.

Reminder

spaces	balls	completeness property	equiv. to
ultrametric spaces	all ultrametric balls	spherically complete	$\mathbf{S}_1$
metric spaces	metric balls with radii in suitable sets of positive real numbers	complete	$\mathbf{S}_1$
ordered abelian groups and fields	all intervals $[a,b]$ with $a \leq b$	symmetrically complete	$\mathbf{S}_1$
topological spaces	all nonempty closed sets	compact	$\mathbf{S}_1^c$
posets	final segments $\uparrow a = \{b \mid a \leq b\}$	directed complete	$\mathbf{S}_4^d$
lattices	final segments $\uparrow a$, initial segments $\downarrow a = \{b \mid a \geq b\}$, and intervals $[a,b]$, $a \leq b$	complete	$\mathbf{S}^*$

DCPOs

If $\{a_i \mid i \in I\}$ is a directed system in the poset $(X, <)$, then $\{\uparrow a_i \mid i \in I\}$ is a directed system of balls in its ball space, and vice versa.

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This proves:

Proposition

A poset X is directed complete if and only if its associated ball space $(X, \{\uparrow a \mid a \in X\})$ is $\mathbf{S}_4^d$.

A poset is called pointed if it has a least element $\perp$.

FPTs for DCPOs

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Theorem

- *(Bourbaki–Witt Theorem) Every increasing function $f : X \to X$ on a DCPO X has a fixed point.*

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Theorem

- *(Bourbaki–Witt Theorem) Every increasing function $f : X \to X$ on a DCPO X has a fixed point.*
- *Every order-preserving function $f : X \to X$ on a pointed DCPO X has a fixed point.*

FPTs for DCPOs

We prove this theorem by checking, simultaneously for both cases, that the conditions of our general FPT are satisfied.

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Assume that there is a ball space structure $(X, \mathcal{B}^f)$ on X for which the following conditions are satisfied:

(1) *every $B \in \mathcal{B}^f$ is f-closed,*

(2) *every $B \in \mathcal{B}^f$ contains a fixed point or some smaller ball $B' \in \mathcal{B}^f$,*

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Then f admits a fixed point in every ball.

Proof

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Set $s = \sup\{x_i \mid i \in I\}$. Then $\bigcap \mathcal{N} = \uparrow s$, with $\uparrow s \in \mathcal{B}^f$ because

$fs \geq s$

$fs \geqslant fx_i \geqslant x_i \Rightarrow fs \geqslant \sup\{x_i\} = s$

When does a DCPO have an S^* ball space?

The ball space associated with a directed complete poset is not always an S^* ball space.

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We have proved:

Proposition

A poset is directed complete and bounded complete if and only if the ball space defined by the final segments $\uparrow a$ is S^* .

Complete lattices

A poset $(X, <)$ is a complete lattice if and only if it is a complete upper semilattice (every subset has a supremum) and a complete lower semilattice (every subset has an infimum).

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Similarly, if a poset has a bottom element, then in the reverse ordering, all subsets are bounded.

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A poset $(X, <)$ is a complete lattice if and only if it is a complete upper semilattice (every subset has a supremum) and a complete lower semilattice (every subset has an infimum).
If a poset has a top element, then all subsets are bounded. If its ball space is $\mathbf{S}^*$, then it is bounded complete and all subsets have a supremum.
Similarly, if a poset has a bottom element, then in the reverse ordering, all subsets are bounded. If the ball space $(X, \{\downarrow a \mid a \in X\})$ is $\mathbf{S}^*$, then X with the reverse ordering is bounded complete and all subsets have a minimum.

Complete lattices

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A poset $(X, <)$ with top and bottom element is a complete lattice if and only if the ball spaces

$$\{\uparrow a \mid a \in X\} \quad \textit{and} \quad \{\downarrow a \mid a \in X\}$$

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Note that the ball space

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is. In the presence of top and bottom, it is equal to $\{[a,b] \mid a,b \in X,\, a \leq b\}$.

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Take a complete lattice $(X, <)$ and a centered system $\{[a_i, b_i] \mid i \in I\}$ of nonempty intervals.

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This is a ball if and only if $\sup_i a_i \leq \inf_i b_i$. Suppose this were not the case. Then there would be some j with $a_j > \inf_i b_i$ and there would be some k with $a_j > b_k$.

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This is a ball if and only if $\sup_i a_i \leq \inf_i b_i$. Suppose this were not the case. Then there would be some j with $a_j > \inf_i b_i$ and there would be some k with $a_j > b_k$. So $[a_j, b_j] \cap [a_k, b_k] = \emptyset$,

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Theorem

A poset $(X, <)$ with top and bottom element is a complete lattice if and only if the ball space $(X, \{[a, b] \mid a, b \in X,\ a \leq b\})$ is $\mathbf{S}^*$.

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A lattice $(X, <)$ is complete if and only if the ball space $(X, \{\uparrow a\,, \downarrow b\,, [a,b] \mid a,b \in X,\, a \leq b\})$ is $\mathbf{S}^*$.

Summary of $\mathbf{S}^*$ ball spaces

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- A poset is a complete lattice if and only if it has a bottom and a top element and the ball space defined by its nonempty closed intervals $[a, b]$ is S^* .

Spherical closure in $\mathbf{S}^*$ ball spaces

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$\mathrm{scl}_{\mathcal{B}}(S) \in \mathcal{B}$ *is the smallest ball containing* S.

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The Knaster-Tarski FPT

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Is there an analogue for ball spaces?
Can it be used to transfer the Knaster-Tarski FPT to other applications?

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Let $\mathrm{Fix}(f)$ be the set of fixed points of f, and set

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- Take a centered system of balls $\{B_i \mid i \in I\}$ in $\mathcal{B}^f_{\mathrm{Fix}}$.
- $B_i = B_i' \cap \mathrm{Fix}(f)$ for some $B_i' \in \mathcal{B}^f$, so $\mathrm{scl}_{\mathcal{B}^f}(B_i) \cap \mathrm{Fix}(f) = B_i$.

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$$\bigcap_{i\in I} B_i = \bigcap_{i\in I} \mathrm{scl}_{\mathcal{B}^f}(B_i) \cap \mathrm{Fix}(f) = B \cap \mathrm{Fix}(f) \in \mathcal{B}_{\mathrm{Fix}} .$$

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For an order-preserving function f on a complete lattice X take the family

$$\mathcal{B}^f := \{[a,b] \mid f([a,b]) \subseteq [a,b]\}.$$

So condition **(1)** of the previous theorem is met.

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Hence, $\bigcap_{i\in I} B_i \in \mathcal{B}^f$.

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Hence, $\bigcap_{i\in I} B_i \in \mathcal{B}^f$.
By the previous theorem, $(\mathrm{Fix}(f), \mathcal{B}^f_{\mathrm{Fix}})$ is an S* ball space.

Proof of Knaster-Tarski FPT

In order to show that the poset $\mathrm{Fix}(f)$ of fixed points is a complete lattice, we will apply the characterization of posets that are complete lattices which we have presented earlier. For this, we have to show two things:

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We will show the second assertion first.

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- For every such c, $[c, \top] \in \mathcal{C}$, so $[b, \top] \subseteq [c, \top]$ and $b = c$.
- It follows that b is the largest fixed point in X.
- Similarly, one finds the smallest fixed point.

Proof of Knaster-Tarski FPT

Now we show that the ball space $(\mathrm{Fix}(f), \{B \cap \mathrm{Fix}(f) \mid B \in \mathcal{B}^f\})$ is equal to

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Since $B = f(B) \subseteq [fa, fb] \subseteq [a,b]$ and $[fa,fb]$ is f-closed, but $[a,b]$ is the smallest ball in $\mathcal{B}^f$ containing B, we have that $fa = a$ and $fb = b$.

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$$(\mathrm{Fix}(f), \{\uparrow a\,, \downarrow b\,, [a,b] \mid a,b \in X,\, a \leq b\})$$

which in turn is equal to

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because we have already shown that $\mathrm{Fix}(f)$ has a top and a bottom element.
Take $B \in \mathcal{B}^f_{\mathrm{Fix}}$. Then $\mathrm{scl}_{\mathcal{B}^f}(B) = [a,b]$ for some $a,b \in X$.

Since $B = f(B) \subseteq [fa, fb] \subseteq [a,b]$ and $[fa,fb]$ is f-closed, but $[a,b]$ is the smallest ball in $\mathcal{B}^f$ containing B, we have that $fa = a$ and $fb = b$.
Thus $B = \mathrm{scl}_{\mathcal{B}^f}(B) \cap \mathrm{Fix}(f) = \{c \in \mathrm{Fix}(f) \mid a \leq c \leq b\}$ with $a,b \in \mathrm{Fix}(f)$.

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This proves that $\mathrm{Fix}(f)$ is a complete lattice.

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Take a spherically complete ultrametric space (X, u) and a nonexpanding function $f : X \to X$ which is contracting on orbits. Then every f-closed ultrametric ball and in particular every ball of the form $B(x, y)$ contains a fixed point, and $(\mathrm{Fix}(f), u)$ is again a spherically complete ultrametric space.

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Theorem

Take a quasi-compact topological space X and a function $f : X \to X$. Assume that every closed, f-closed set contains a fixed point or a smaller closed, f-closed set. Then the topology on the set $\mathrm{Fix}(f)$ of fixed points of f having $\mathcal{B}^f \cap \mathrm{Fix}(f)$ as its collection of closed sets is itself quasi-compact.

An open question for topologists

As we are rather interested in the topology on $\mathrm{Fix}(f)$ induced by the original topology of X, we ask:

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Open question: Give a criterion on f which guarantees that

$$\mathcal{B}^f \cap \mathrm{Fix}(f) \;=\; \mathcal{B} \cap \mathrm{Fix}(f)\,.$$

The Tychonoff theorem for ball spaces

Given a collection of ball spaces $(X_j, \mathcal{B}_j)_{j\in J}$, we define their (box) product by setting

$$\mathcal{B} := \left\{ \prod_{j\in J} B_j \mid \forall j \in J : B_j \in \mathcal{B}_j \right\}.$$

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Take **S** *to be any of the properties in the hierarchy of spherical completeness.The product* $(\prod_{j \in J} X_j, \mathcal{B})$ *has property* **S** *if and only if all ball spaces* $(X_j, \mathcal{B}_j)$*,* $j \in J$*, have property* **S**.

Sketch of proof

Take ball spaces $(X_j, \mathcal{B}_j)$, $j \in J$, and in every $\mathcal{B}_j$ take a set of balls $\{B_{i,j} \mid i \in I\}$.

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As "nonempty intersection" can be replaced by "contains a ball", "contains a largest ball" or "is a ball", a similar argument shows that also the properties $\mathbf{S}_2$, $\mathbf{S}_3$ and $\mathbf{S}_4$ are transfered.

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$$\emptyset \neq \bigcap_{i\in I} \prod_{j\in J} B_{i,j} = \left(\bigcap_{i\in I} B_i\right) \times \left(\prod_{j_0\neq j\in J} B^j\right),$$

whence $\bigcap_{i\in I} B_i \neq \emptyset$. We have proved that for every $j \in J$, $(X_j, \mathcal{B}_j)$ is spherically complete.

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A similar proof works for the properties $\mathbf{S}_2$, $\mathbf{S}_3$ and $\mathbf{S}_4$.

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For the transfer of the other properties, observe that $\{\prod_{j\in J} B_{i,j} \mid i \in I\}$ will be a directed system if and only if all sets $\{B_{i,j} \mid i \in I\}$, $j \in J$, are.

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The same holds for "centered system" in place of "directed system".

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However, the stronger properties of the hierarchy may get lost under this restriction.

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This shows that the product is the ultrametric ball space for the product ultrametric on $\prod_{j\in J} X_j$ which is defined as

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The latter is a poset, but in general not totally ordered, even if all Γ_j are and even if J is finite. So the product ultrametric is a natural example for an ultrametric with partially ordered value set.

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with $\gamma \in \bigcup_{j\in J}\Gamma_j$. Now the value set is totally ordered. It turns out that the collection of balls so obtained is a (usually proper) subset of the full ultrametric ball space of the product ultrametric. Therefore, if all (X_j, u_j) are spherically complete, then so is $(\prod_{j\in J} X_j, u_{\max})$.

Finite products of ultrametric spaces: an appliction

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When the index set J is infinite, do we still have the chance to obtain an ultrametric on the product with a totally ordered value set?

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In which way does Tychonoff's theorem follow from its analogue for ball spaces? The problem in the case of topological spaces is that the product ball space we have defined, while containing only closed sets of the product, does not contain all of them, as it is not necessarily closed under finite unions and arbitrary intersections. If we close it under these operations, are its spherical completeness properties maintained?

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In order to prove this theorem, we need a lemma that is inspired by Alexander's Subbase Theorem:

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In order to prove this theorem, we need a lemma that is inspired by Alexander's Subbase Theorem:

Lemma

If $\mathcal{S}$ is a maximal centered system of balls in $\mathcal{B}'$ (that is, no subset of $\mathcal{B}'$ properly containg $\mathcal{S}$ is a centered system), then there is a subset $\mathcal{S}_0$ of $\mathcal{S}$ which is a centered system in $\mathcal{B}$ and has the same intersection as $\mathcal{S}$.

Proof of the lemma

It suffices to prove the following: if $B_1, \ldots, B_n \in \mathcal{B}$ such that $B_1 \cup \ldots \cup B_n \in \mathcal{S}$,

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Suppose that $B_1, \ldots, B_n \in \mathcal{B} \setminus \mathcal{S}$. By the maximality of $\mathcal{S}$ this implies that for each $i \in \{1, \ldots, n\}$, $\mathcal{S} \cup \{B_i\}$ is not centered. This in turn means that there is a finite subset $\mathcal{S}_i$ of $\mathcal{S}$ such that $\bigcap \mathcal{S}_i \cap B_i = \emptyset$. But then $\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n$ is a finite subset of $\mathcal{S}$ such that

$$\bigcap(\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n) \cap (B_1 \cup \ldots \cup B_n) \;=\; \emptyset\,.$$

This yields that $B_1 \cup \ldots \cup B_n \notin \mathcal{S}$, which proves our assertion.

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Write $B_i = \bigcap_{j \in J_i} B_{i,j}$ with $B_{i,j} \in \mathcal{B}$.

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Closure under unions and intersections

From the previous two theorems we obtain:

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Note that since $(X, \mathcal{B}')$ is an $\mathbf{S}_1^c$ ball space by the previous two theorems and is intersection closed, it follows that it is an $\mathbf{S}^*$ ball space.

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If $\mathcal{B} \subseteq \mathcal{B}'$ and $(X, \mathcal{B}')$ is an $\mathbf{S}_1^c$ ball space, then so is $(X, \mathcal{B})$. The same holds for $\mathbf{S}_1$ and $\mathbf{S}_1^d$ in place of $\mathbf{S}_1^c$.

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Which are the topologies we obtain in this way?

Example: the p-adics

The field $\mathbb{Q}_p$ of p-adic numbers together with the p-adic valuation v_p is spherically complete.

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as basic open sets. It turns out that this topology is finer than the one we derived from the ball space.

Unions of ball spaces (“hybrid” ball spaces)

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Open problem: What sbout directed and centered systems in unions of ball spaces?

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If we join them, we obtain an $\mathbf{S}_2$ *ball space.*

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Observe that this ball space is contained in the ball space obtained from $\mathcal{B}$ by closing under finite unions. But as $\mathcal{B}$ is not $\mathbf{S}_1^c$, we cannot apply our theorem about finite unions.

Pulling nests back and forth

Take two ball spaces $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ and a function $f : X \to X'$.

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Lemma

a) If f is ball continuous and $\mathcal{N}'$ is a nest of balls in $\mathcal{B}'$, then the preimages of the balls in $\mathcal{N}'$ form a nest of balls in $\mathcal{B}$.
b) If f is ball closed and $\mathcal{N}$ is a nest of balls in $\mathcal{B}$, then the images of the balls in $\mathcal{N}$ form a nest of balls in $\mathcal{B}'$.

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b) If f is ball continuous, ball closed and surjective, then the posets $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic and all spherical completeness properties transfer in both directions.

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Corollary

If $(X', \mathcal{B}')$ is a quotient space of $(X, \mathcal{B})$ and if $(X, \mathcal{B})$ is spherically complete, then so is $(X', \mathcal{B}')$.

Reminder: the Ultrametric Attractor Theorem

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The latter property may give the right idea for the definition of "spherically continuous" functions for ultrametric spaces as well as for ball spaces — new work in progress with Katarzyna and Rene Bartsch (TU Darmstadt).

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$$u(s,t) \;:=\; \{r \in T \mid r \leq s \text{ if and only if } r \leq t\} \in \mathcal{P}(T)\,.$$

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With respect to the order on $\mathcal{P}(T)$ defined by reverse inclusion, u is an ultrametric on T. Its value set is $\mathcal{P}(T)$, with least element T.

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Suppose there is a subset $T_0 \subset T$ such that every element in T is the supremum of a subset of T_0. Then for each $s, t \in T$, we set

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With respect to the order on $\mathcal{P}(T_0)$ defined by reverse inclusion, u is an ultrametric on T with its value set contained in $\mathcal{P}(T_0)$.

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$$\min\{\alpha \leq \beta \quad | \quad r \leq s \text{ if and only if } r \leq t \\ \text{for every } r \in T_0 \text{ with } \varphi(r) < \alpha\}\,.$$

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This is an ultrametric on T. Its value set is the ordinal $\beta + 1 = \beta \cup \{\beta\}$, endowed with the reverse ordering and having β as its least element.

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Because of the first condition, there is an ultrametric on T with values in $\mathcal{P}(C)$. In order to obtain an ultrametric u_φ with a totally ordered value set, one usually takes a rank function φ from C into a countable ordinal β.

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Note that by taking $Y = X$ *and* g *to be the identity function we obtain the General FPT for ball spaces.*

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- Thus $f(x) = g(x) = y$ for every $x \in \bigcap_{B\in\mathcal{N}_0} B \subseteq B_0$.

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- By the maximality of $\mathcal{N}_0$ and (CS2) we obtain that $f(B)$ or $g(B)$ is a singleton $\{y\}$ for some $y \in Y$.
- By (CS1), $f(B) \cap g(B) = \{y\}$, so for some $x \in B \subseteq B_0$, $fx = gx = y$.

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Preprints and further information

The Valuation Theory Home Page
http://math.usask.ca/fvk/Valth.html