MODEL THEORY OF TAME VALUED FIELDS AND BEYOND: RECENT DEVELOPMENTS AND OPEN QUESTIONS

FRANZ-VIKTOR KUHLMANN

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This talk was a survey on the progress of the model theory of valued fields since my work on tame fields [9], and the main problems that have remained open. The slides of the talk are available at `https://www.fvkuhlmann.de/Talk-ddg40-2025-nopause.pdf`.

For a valued field (K, v), we denote its value group by vK, its residue field by Kv, and its valuation ring by $\mathcal{O}_K$. A valued field (K, v) is called **henselian** if for each algebraic extension $L|K$ there is a unique extension of v to L. An algebraic extension $(L|K, v)$ of a henselian valued field (K, v) is called **immediate** if the induced embeddings $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lv$ are onto, and it is called **tame** if every finite subextension $K'|K$ satisfies the following conditions:

(T1) the ramification index $(vK' : vK)$ is not divisible by $\mathrm{char}\, Kv$,

(T2) the residue field extension $K'v|Kv$ is separable,

(T3) the extension $(K'|K, v)$ is **defectless**, i.e., $[L : K] = (vL : vK)[Lv : Kv]$.

A henselian valued field (K, v) is called a **tame field** if the algebraic closure K^{ac} of K with the unique extension of v is a tame extension of (K, v).

The perfect hull $\mathbb{F}_p((t))^{1/p^\infty}$ of the field $\mathbb{F}_p((t))$ of formal Laurent series over the field $\mathbb{F}_p$ with p elements is perfect but not tame, as it admits an immediate extension of degree p, violating (T3).

For details on tame fields, see [9]. The results of this paper are now frequently applied in the model theory of valued fields. In particular:

Theorem 1 (Kuhlmann (2016))**.** *Tame fields (K, v) satisfy model completeness in the language $\mathcal{L}_{\mathrm{val}}$ of valued rings relative to the elementary theories of their value groups vK in the language of ordered groups and their residue fields Kv in the language of rings. If* $\mathrm{char}\, K = \mathrm{char}\, Kv$*, then also relative completeness and relative decidability hold.*

However, there are still daunting questions about tame fields that have remained unanswered. For instance,

• **Open problem:** Do tame fields admit quantifier elimination in a suitable language?

There are also open problems concerning relative completeness and decidability for tame fields (K, v) that have **mixed characteristic**, i.e., $\mathrm{char}\, K = 0$ while $\mathrm{char}\, Kv = p > 0$; see [4]. Progress on these problems has been made in [12].

In [9], the relative decidability proven for tame fields of positive characteristic is applied to deduce:

Theorem 2 (Kuhlmann (2016)). *Take $q = p^n$ for some prime p and some $n \in \mathbb{N}$, and an ordered abelian group Γ. Assume that Γ is divisible or elementarily equivalent to the p-divisible hull of $\mathbb{Z}$. Then the $\mathcal{L}_{\mathrm{val}}$-elementary theory of the power series field $\mathbb{F}_q((t^\Gamma))$ with coefficients in $\mathbb{F}_q$ and exponents in Γ, endowed with its canonical valuation v_t, is decidable.*

In [12], Lisinski generalizes this result to a version using the language $\mathcal{L}_{\mathrm{val}}(t)$, i.e., the language $\mathcal{L}_{\mathrm{val}}$ enriched by a constant symbol t.

Since Ax and Kochen, and independently Ershov, established in 1965 the decidability of the elementary theory of the field $\mathbb{Q}_p$ of p-adic numbers, several questions about the decidability of the elementary or the existential theory of local fields and their extensions have been answered, and several others have remained open. We have already seen some results in equal positive charateristic. In contrast, less is known in mixed characteristic, for instance about

– the totally ramified extension $\mathbb{Q}_p(\zeta_{p^\infty})$ obtained from $\mathbb{Q}_p$ by adjoining all p^n-th roots of unity, $n \in \mathbb{N}$,

– the totally ramified extension $\mathbb{Q}_p(p^{1/p^\infty})$ obtained from $\mathbb{Q}_p$ by adjoining a compatible system of p^n-th roots of p, $n \in \mathbb{N}$,

– the maximal abelian extension $\mathbb{Q}_p^{ab}$ of $\mathbb{Q}_p$.

These are studied in [7].

Theorem 3 (Kartas (2024)). *The fields $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p(p^{1/p^\infty})$ equipped with their unique extensions v_p of the p-adic valuation admit maximal immediate extensions which have decidable elementary $\mathcal{L}_{\mathrm{val}}$-theories.*

• **Open problem:** Are $\mathbb{Q}_p(\zeta_{p^\infty})$, $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p^{ab}$ decidable in $\mathcal{L}_{\mathrm{val}}$? Are $\mathbb{F}_p((t))^{1/p^\infty}$ and $\mathbb{F}_p^{\mathrm{ac}}((t))^{1/p^\infty}$ decidable in $\mathcal{L}_{\mathrm{val}}$ or even $\mathcal{L}_{\mathrm{val}}(t)$?

Theorem 4 (Kartas (2024)). *(a) If $\mathbb{F}_p((t))^{1/p^\infty}$ has a decidable elementary or existential $\mathcal{L}_{\mathrm{val}}(t)$-theory, then $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p(p^{1/p^\infty})$ have decidable elementary or existential $\mathcal{L}_{\mathrm{val}}$-theories, respectively.*
(b) If $\mathbb{F}_p^{\mathrm{ac}}((t))^{1/p^\infty}$ has a decidable elementary or existential $\mathcal{L}_{\mathrm{val}}(t)$-theory, then $\mathbb{Q}_p^{ab}$ has a decidable elementary or existential $\mathcal{L}_{\mathrm{val}}$-theory, respectively.

• **Open problem:** What about the reverse direction?

A henselian valued field (K, v) is a **separably tame field** if the separable-algebraic closure of K is a tame extension of (K, v). With suitable enrichments of the language $\mathcal{L}_{\mathrm{val}}$, Theorem 1 has been generalized to separably tame fields of positive characteristic p with $[K : K^p] < \infty$ in [10]; the latter condition has been removed in [1].

Peter Scholze defines a **perfectoid field** to be a complete nondiscrete rank 1 valued field of positive residue characteristic such that the Frobenius is surjective on $\mathcal{O}_K/p\mathcal{O}_K$. Neither "complete" nor "rank 1" are elementary properties. A suitable elementary class of valued fields containing the perfectoid fields is that of deeply ramified fields, studied in the article [11]. A nontrivially valued field (K, v) is **deeply ramified** if and only if the following conditions hold:

(DRvg) whenever $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of the value group vK, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$ (that is, no archimedean component of vK is discrete);

(DRvr) if $\mathrm{char}\, Kv = p > 0$, then the Frobenius is surjective on $\mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}}$, where $\hat{K}$ denotes the completion of (K, v).

Every perfect valued field of positive characteristic and every tame field is deeply ramified.

Inspired by the notion "roughly p-divisible value group" introduced by Will Johnson, we call (K, v) a **roughly deeply ramified field** if it satisfies axiom (DRvr) together with:
(DRvp) if char $Kv = p > 0$, then $v(p)$ is not the smallest positive element in the value group vK.
The two axioms (DRvp) and (DRvr) together imply that the smallest convex subgroup of vK containing $v(p)$ if char $K = 0$, and vK itself if char $K = p$, is p-divisible.

In the article [6], Jahnke and Kartas develop a model theoretic approach to the tilting construction. They work with an elementary class $\mathcal{C}$ of henselian fields (K, v) of residue characteristic $p > 0$ with distinguished element $\pi \in K \setminus \{0\}$, $v\pi > 0$, such that:
the Frobenius on the ring $\mathcal{O}_K/(p)$ is surjective, and with the coarsening w of v associated with the valuation ring $\mathcal{O}_v[\pi^{-1}]$, (K, w) is **algebraically maximal**, i.e., does not admit immediate algebraic extensions.

The class $\mathcal{C}$ contains all henselian roughly deeply ramified fields of mixed characteristic. For this class, Jahnke and Kartas prove analogues of the model theoretic results for tame fields, with the residue fields Kv replaced by the residue rings $\mathcal{O}_K/(\pi)$. This "mods out" the non-tame part of the valued fields in $\mathcal{C}$. So we are still left with the
• **Open problem:** What can we say about the model theory of (roughly) deeply ramified fields, and in particular of $\mathbb{F}_p((t))^{1/p^\infty}$?

It is well known that the henselization $\mathbb{F}_p(t)^h$ of $\mathbb{F}_p(t)$ is existentially closed in $\mathbb{F}_p((t))$. However, the following has remained a daunting
• **Open problem:** Is $\mathbb{F}_p(t)^h$ an elementary substructure of $\mathbb{F}_p((t))$?
In contrast, Jahnke and Kartas prove that $\mathbb{F}_p(t^{1/p^\infty})^h$ is an elementary substructure of $\mathbb{F}_p((t))^{1/p^\infty}$. This positive result encourages us to ask:
• **Open problem:** Is it possible to prove model theoretic results for henselian perfect valued fields of positive characteristic, analoguous to those for tame fields (but under mild additional conditions)?

The following is shown in the article [3]:
Theorem 5 (Anscombe – Fehm (2016)). *The existential $\mathcal{L}_{\mathrm{val}}$-theory of $\mathbb{F}_p((t))$ is decidable.*
However, as can be seen from our discussion of Kartas' work, we would like to have more. Jan Denef and Hans Schoutens proved in 2003 that the existential $\mathcal{L}_{\mathrm{val}}(t)$-theory of $\mathbb{F}_p((t))$ is decidable, provided that resolution of singularities holds in positive characteristic. In order to discuss more recent improvements of this result, some preparations are needed. In the article [8] the following question is studied: Take a field extension $F|K$ such that F admits a K-rational place, or in other words, a valuation with residue field K. Under which additional conditions does it follow that K is existentially closed in F? Here a key role is played by large fields. While they are usually defined in a different way (see [8, Section 1.3]), one can also use the model theoretic approach: A field K is **large** if it is existentially closed in $K((t))$.

Theorem 6 (Kuhlmann (2004)). *Let K be a perfect field. Then the following conditions are equivalent:*
1) K is a large field,
2) K is existentially closed in every power series field $K((t^\Gamma))$,
3) K is existentially closed in every extension field L which admits a K-rational place.

Local uniformization is a local form, and a consequence, of resolution of singularities.

Theorem 7 (Kuhlmann (2004))**.** *If all rational places of arbitrary function fields admit local uniformization, then the three conditions of the previous theorem are equivalent, for arbitrary fields K.*

In the paper [2] the assumption that implication 1)⇒3) holds for *arbitrary* fields K is called hypothesis **(R4)**. Hence local uniformization implies (R4). The authors prove:

Theorem 8 (Anscombe – Dittmann – Fehm (2023))**.** *If (R4) holds, then the existential $\mathcal{L}_{\rm val}(t)$-theory of $\mathbb{F}_p((t))$ is decidable.*

• **Open problem:** Does (R4) hold?

A longstanding open problem is whether the elementary theory of $\mathbb{F}_p((t))$ is decidable. In [5] it is shown that $(\mathbb{F}_p((t)), v_t)$ is **extremal**, i.e., if for every multi-variable polynomial $f(X_1, \ldots, X_n)$ over K, the set $\{v(f(a_1, \ldots, a_n)) \mid a_1, \ldots, a_n \in \mathbb{O}_K\} \subseteq vK \cup \{\infty\}$ has a maximal element.

• **Open problem:** Is the axiom system "extremal valued field of positive characteristic with value group elementarily equivalent to $\mathbb{Z}$ and residue field $\mathbb{F}_p$" complete?

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Institute of Mathematics, University of Szczecin, ul. Wielkopolska 15, 70-451 Szczecin, Poland

Email address: `fvk@usz.edu.pl`