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Abelian groups with contractions I

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ABSTRACT. On the value group of nonarchimedean exponential fields with respect to their natural valuation, the exponential induces a map called contraction. A corresponding theory of abelian groups with contractions is axiomatized and, by a detailed study of their algebraic properties, shown to be model complete, complete, decidable and to admit quantifier elimination. Finally, other concepts of contractions are discussed.

1. Introduction

In the last years, the model theory of the reals with exponentiation has gone through a remarkable progress by the works of A. Wilkie and other authors (cf. [W], [D–M–M]). This has increased the interest in the structure of ordered fields with an exponential map, in particular if they are (at least in part) models of the above mentioned theory. In [KS] and [K–KS], the structure of nonarchimedean exponential fields and in particular, of their value groups with respect to the natural valuation, has been studied. It has turned out that an exponential induces a map χ, called *contraction*, on the value group. It is essentially given as follows: if v is the natural valuation and x a field element with $vx < 0$, then $\chi vx = v \log x$, where log is the inverse of the exponential. The contraction has some amazing properties: although "contracting" archimedean classes, it is still surjective. For the discussion how to obtain contractions from exponentials, see chapter 3 of [K–KS]. In the present paper, we will take over the axiom system that was derived there to study the model theoretic properties of divisible ordered abelian groups with centripetal contractions. The property "centripetal" corresponds to a growth axiom satisfied by the usual exponential.

In section 2, we will give some preliminary facts about ordered abelian groups and their natural valuations.

At the beginning of section 3, the axioms for contractions on ordered abelian groups will be listed and discussed. Afterwards, we study precontraction groups (on which the contraction is not assumed to be surjective) and their extensions. By means of embedding lemmas, the model theory of divisible centripetal contraction

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groups will be shown to be model complete, complete, decidable and to admit quantifier elimination (Theorems 3.23, 3.26, 3.27 and 3.24).

In section 4, we will finally discuss some other possible concepts for contractions on (not necessarily ordered) abelian groups.

In a forthcoming paper [K2], we will study the terms built up with contractions, obtaining a description of the definable sets. This description together with the quantifier elimination result of the present paper will show that the theory of divisible centripetal contraction groups is weakly o-minimal.

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2. Preliminaries

In this paper, we will assume $\mathbb{N}$, the set of natural numbers, to contain 0. Further, whenever we will talk of ordered groups, we will mean ordered abelian groups.

Let G be an ordered group. We write $G^{<0} = \{g \in G \mid g < 0\}$ and $G^{>0} = \{g \in G \mid g > 0\}$. We set $\mathrm{sign}(0) = 0$ and for $a \in G$, we set $\mathrm{sign}(a) = 1$ if $a > 0$, and $\mathrm{sign}(a) = -1$ if $a < 0$. Further, we set $|a| := \max\{a, -a\} = \mathrm{sign}(a) \cdot a$. Two elements $a, b \in G$ are called *archimedean equivalent* if there is some $n \in \mathbb{N}$ such that $n|a| \geq |b|$ and $n|b| \geq |a|$. Let va denote the equivalence class of a. The set of equivalence classes is ordered as follows: $va < vb$ if and only if $|a| > |b|$ and $va \neq vb$. We write $\infty := v0$; this is the maximal element in the ordered set of equivalence classes. The map $a \mapsto va$ is a group valuation, that is, it satisfies

(V0) $\forall x:\ vx = \infty \Leftrightarrow x = 0\,,$

(V1) $\forall x, y:\ v(x-y) \geq \min\{vx, vy\}\,.$

From these rules, we may deduce

(V2) $\forall x:\ vx = v(-x)\,,$

(V3) $v(\sum_{1\leq i\leq n} x_i) = \min_{1\leq i\leq n} vx_i$ if all nonzero x_i have different values,

(V4) $\forall x, y:\ v(x-y) > \min\{vx, vy\} \Rightarrow vx = vy\,.$

The ordered set $vG := \{vg \mid 0 \neq g \in G\}$ will be called the *value set* of G.

The valuation that we have defined above, is canonically associated to every ordered group and is called the *natural valuation*. The order type of vG is called the *rank* of the ordered group G. Hence, G is archimedean if and only if its rank is 1, that is, vG consists of just one element. Note that for every $a \in G$ and every $n \in \mathbb{Z} \setminus \{0\}$, the element $na \in G$ is archimedean equivalent to a and so, the natural valuation satisfies the axiom scheme

(NV1) $\forall x:\ v(nx) = vx \qquad (0 \neq n \in \mathbb{Z})\,.$

Further, the natural valuation is compatible with the ordering, in the following sense:

(NV2) $\forall x, y \in G^{<0}:\ (vx < vy \Rightarrow x < y) \wedge (x < y \Rightarrow vx \leq vy).$

Let us also note:

(NV3) $\mathrm{sign}(\sum\limits_{1\leq i\leq n} x_i) = \mathrm{sign}(x_m)$ if $vx_m < vx_i$ for all $i \neq m$,

(NV4) $\forall x, y:\ v(x-y) > vx \Rightarrow \mathrm{sign}(x) = \mathrm{sign}(y).$

Further, (NV3) may be generalized to

(NV5) $x_m < x'_m \Rightarrow \sum_{1\leq i\leq n} x_i < \sum_{1\leq i\leq n} x'_i$ if $vx_m < vx_i$ and $vx'_m < vx'_i$ for all $i \neq m$.

In the sequel, v will always denote the natural valuation, if not stated otherwise.

Given an ordered Hahn sum $\coprod_{i\in I} A_i$ or Hahn product $\mathbf{H}_{i\in I} A_i$ over archimedean ordered groups A_i, then the ordered index set I is order isomorphic to the ordered set of archimedean classes, and via this isomorphism, the natural valuation is (equivalent to) the minimum support valuation, that is, for $a = (a_i)_{i\in I}$ we may set

$$va = \min\{i \in I \mid a_i \neq 0\} \ .$$

Let $(S,<)$ be a totally ordered set. If $S_1, S_2 \subset S$ and $a \in S$, we will write $a < S_2$ if $a < b$ for all $b \in S_2$, and further, we will write $S_1 < S_2$ if $a < S_2$ for all $a \in S_1$. Similarly, we use the relations $>$, $\leq$ and $\geq$. A pair (S_1, S_2) of two convex subsets of S satisfying $S_1 \cup S_2 = S$ will be called a *quasicut in S* if $S_1 \leq S_2$, and it will be called a *cut* if it even satisfies $S_1 < S_2$. We allow S_1 or S_2 to be empty, with the convention that $\emptyset < G$ and $G < \emptyset$. We will say that $a \in S$ *realizes the (quasi)cut* (S_1, S_2) if $S_1 \leq a \leq S_2$. If the ordered set T contains S and if $b \in T$, then $(\{a \in S \mid a \leq b\}, \{a \in S \mid a > b\})$ will be called the *cut induced by b in S*. Note that a (quasi)cut in a densely ordered set can be realized by at most one element of that set.

If (G_1, G_2) is a quasicut in G and $g \in G$, then

$$(G_1, G_2) - g \ := \ (\{a - g \mid a \in G_1\}\,,\, \{a - g \mid a \in G_2\})$$

is again a quasicut in G, called a *shift* of (G_1, G_2). If (G_1, G_2) is a cut, then also $(G_1, G_2) - g$ is a cut. If (G_1, G_2) is the cut induced by b in G, then $(G_1, G_2) - g$ is the cut induced by $b - g$ in G. Note that two elements $a \neq a'$ may determine the same shift $(G_1, G_2) - a = (G_1, G_2) - a'$.

To every quasicut (G_1, G_2) in G, we may associate a quasicut in vG in the following way. If G_2 contains an element ≤ 0, then we set $S_1 = v(G_1)$ and $S_2 = v(G^{<0} \cap G_2)$ and find that (S_1, S_2) is a quasicut in vG by virtue of (NV2). If on the other hand, G_2 is contained in $G^{>0}$, then $-G_1$ contains an element ≤ 0, and we set $S_1 = v(-G_2) = v(G_2)$ and $S_2 = v(G^{<0} \cap -G_1) = v(G^{>0} \cap G_1)$; again, (S_1, S_2) is a quasicut in vG. If (S_1, S_2) is a cut, then (G_1, G_2) will be called a *v-cut*. From this definition, we may deduce the following criterion: (G_1, G_2) is a v-cut if and only if there is no pair (g_1, g_2) of elements $g_1 \in G_1$ and $g_2 \in G_2$ such that $vg_1 = vg_2$ and $\mathrm{sign}(g_1) = \mathrm{sign}(g_2)$. Using this criterion together with (NV2), we see that every element b in an extension of $(G,<)$ with $vb \notin vG \cup \{\infty\}$ will induce a v-cut in G. There is also a converse, stated as part b) of the following lemma:

LEMMA 2.1. *Let G be divisible and (G_1, G_2) be a cut in G.*
a) Among all shifts of (G_1, G_2), there is at most one v-cut. If (G_1, G_2) is realized by some element $a \in G$, then $(G_1, G_2) - a$ is this unique v-cut, realized by 0.
b) Assume (G_1, G_2) to be a v-cut which is not realized in G. If $(H,<)$ is an extension of $(G,<)$ and $b \in H$ realizes (G_1, G_2), then $vb \notin vG \cup \{\infty\}$.

Proof: a): Suppose that $(G_1, G_2) - g$ and $(G_1, G_2) - g'$ are v-cuts; we have to show that they are equal. Replacing G_i by $G_i - g' = \{h - g' \mid h \in G_i\}$, $i = 1, 2$, and g by $g-g'$, we may assume from the start that $g'=0$. Further, let us assume that $-g\in G_1$; for $-g\in G_2$, the proof is symmetrical.

Suppose that $(G_1-g)\cap G_2\neq\emptyset$ and choose $a\in G_1$ such that $a-g\in G_2$. If $a\geq -g$, it follows that $2a\in G_2$ since $2a\geq a-g$. If $a\leq -g$, it follows that $-2g\in G_2$ since $-2g\geq a-g$. In view of $a,-g\in G_1$, both is impossible by our criterion for v-cuts.

Now suppose that $G_1\cap(G_2-g)\neq\emptyset$ and choose $b\in G_2$ such that $b-g\in G_1$. Then $b-2g=(b-g)-g\in G_1-g$. Applying our criterion to (G_1,G_2), we find that also $b/2\in G_2$ and thus, $b/2-g\in G_2-g$. Applying our criterion to $(G_1,G_2)-g$, we obtain $b-2g=2(b/2-g)\in G_2-g$, contradicting $b-2g\in G_1-g$.

We have now shown that (G_1,G_2) and $(G_1,G_2)-g$ are equal. As to the last assertion of a), we leave it to the reader to show that $(G_1,G_2)-a$ is a v-cut realized by 0. The uniqueness then follows from what we have already proved.

b): Assume that b realizes (G_1,G_2) and that there is some $a\in G$ such that $va=vb$, that is, a and b are archimedean equivalent. Since G is assumed to be divisible, this implies the existence of some $q\in\mathbb{Q}\setminus\{0\}$ such that b lies between a and $qa\in G$. Since a and qa are archimedean equivalent, we have $va=vqa$ and our above criterion shows that (G_1,G_2) cannot be a v-cut. □

Note that the proof shows that instead of requiring G to be divisible, it suffices to suppose that G is p-divisible for some prime p (which need not be equal to 2). Without this divisibility condition, the lemma is not even true for densely ordered groups. Indeed, in the lexicographic product $\mathbb{Z}\amalg\mathbb{Q}$, both

$$(\{(x,y)\mid x\leq 0\},\{(x,y)\mid x\geq 1\})$$

and

$$(\{(x,y)\mid x\leq -1\},\{(x,y)\mid x\geq 0\})=(\{(x,y)\mid x\leq 0\},\{(x,y)\mid x\geq 1\})\,-\,(1,0)$$

are v-cuts.

The cut (G_1,G_2) will be called a *shifted v-cut* if there exists some $g\in G$ such that $(G_1,G_2)-g$ is a v-cut. We have:

Corollary 2.2. *Let the situation be as in the previous lemma. Assume that (G_1,G_2) is not realized in G but is realized by $b\in H$, where $(H,<)$ is an extension of $(G,<)$. Then (G_1,G_2) is a shifted v-cut if and only if there is some $g\in G$ such that $v(b-g)\notin vG\cup\{\infty\}$. In this case, $(G_1,G_2)-g$ is the unique v-cut among all shifts of (G_1,G_2).*

Proof: Assume that $(G_1,G_2)-g$ is a v-cut. Since b realizes (G_1,G_2), the element $b-g$ realizes $(G_1,G_2)-g$. Hence by part b) of Lemma 2.1, $v(b-g)\notin vG\cup\{\infty\}$. Conversely, if for some $g\in G$ we have $v(b-g)\notin vG\cup\{\infty\}$, then by our remark preceding Lemma 2.1, $(G_1,G_2)-g$ is a v-cut. The uniqueness assertion follows directly from Lemma 2.1. □

The model theoretical facts that we will use in this paper, can be found in most of the books on general model theory, e.g. [C–K] or [P]. Also, they can be found in [K1]. In particular, let us mention the following facts (cf. Korollar 2.19 of [P], or [K1]):

LEMMA 2.3. *Let $\mathcal{S}'$ be a substructure of the structure $\mathcal{S}$. If $\mathcal{S}'$ is existentially closed in $\mathcal{S}$, then $\mathcal{S}$ is embeddable over $\mathcal{S}'$ in every $|\mathcal{S}|^+$-saturated elementary extension of $\mathcal{S}'$. Conversely, if $\mathcal{S}$ is embeddable over $\mathcal{S}'$ in some elementary extension of $\mathcal{S}'$, then $\mathcal{S}'$ is existentially closed in $\mathcal{S}$.*

Note that $|\mathcal{S}|^+$ denotes the successor cardinal of the cardinality of (the underlying set of) $\mathcal{S}$.

We will need the following well known facts about ordered groups. Their proofs can be found in [K1].

LEMMA 2.4. *Let $(G,<) \subset (H,<)$ be an extension of ordered abelian groups and $b \in H \setminus G$. Assume that G be divisible. If $b' \notin G$ is also an element of some extension of $(G,<)$, inducing in G the same cut as b, then the assignment $b \mapsto b'$ defines an order preserving isomorphism from $G+\mathbb{Z}b$ onto $G+\mathbb{Z}b'$. On the other hand, in every $|G|^+$-saturated extension of $(G,<)$ there is an element $b' \notin G$ inducing in G the same cut as b.*

(The assertions of this lemma can easily be used to show the model completeness of the elementary class of nontrivial divisible ordered abelian groups.)

LEMMA 2.5. *Let $(G,<) \subset (H,<)$ and $(G,<) \subset (H',<)$ be extensions of ordered abelian groups. Suppose that b_i and b'_i, $i \in I$, are elements of H and H' respectively, such that*

1) all vb_i, $i \in I$, are different and not contained in vG,

2) the assignment $vb_i \mapsto vb'_i$ establishes an order isomorphism

$$\tau:\, vG \cup \{vb_i \mid i \in I\} \to vG \cup \{vb'_i \mid i \in I\}\,.$$

Then the assignment $b_i \mapsto b'_i$ defines order and valuation preserving isomorphisms

$$G+\sum_{i\in I}\mathbb{Z}b_i \;\longrightarrow\; G+\sum_{i\in I}\mathbb{Z}b'_i$$

$$G+\sum_{i\in I}\mathbb{Q}b_i \;\longrightarrow\; G+\sum_{i\in I}\mathbb{Q}b'_i$$

both inducing τ. (Here, the groups are endowed with the restrictions of the orders of the divisible hulls of H and H' which are uniquely determined by those of H and H'.) Note that $vG \cup \{vb_i \mid i \in I\}$ is the value set of both groups on the left hand side, and $vG \cup \{vb'_i \mid i \in I\}$ is the value set of both groups on the right hand side.

Alternatively, one may replace 2) by the condition

2') the assignment $b_i \mapsto b'_i$ establishes an isomorphism

$$\iota:\, G \cup \{b_i \mid i \in I\} \to G \cup \{b'_i \mid i \in I\}$$

as ordered sets.

Then ι extends linearly to order and valuation preserving isomorphisms of the above groups.

Note that by virtue of $G \cap \{b_i \mid i \in I\} = \emptyset$, the ordering on $G \cup \{b_i \mid i \in I\}$ is already determined by the cuts induced in G by the elements b_i and by the ordering on the set $\{b_i \mid i \in I\}$.

LEMMA 2.6. *Let $(H,<)$ be generated by n elements over $(G,<)$. Then $vH \setminus vG$ consists of at most n values.*

3. Ordered abelian groups with contractions

Let us now introduce contraction maps. We will work in the language $\mathcal{L}_{\rm cg} = \{+,-,0,<,\chi\}$ where $<$ is a binary relation symbol, $+$ is a binary function symbol, and $-$ and χ are unary function symbols. If $(G,+,-,0,<,\chi)$ is an $\mathcal{L}_{\rm cg}$–structure, then it will be called a *precontraction group* if it satisfies

(OAG) $(G,+,-,0,<)$ is an ordered abelian group,
(C0) $\forall x:\ \chi x = 0 \Leftrightarrow x = 0\,,$
(C$\leq$) χ preserves $\leq$,
(C$-$) $\forall x:\ \chi(-x) = -\chi x\,,$
(CA) if x is archimedean equivalent to y and $\mathrm{sign}(x) = \mathrm{sign}(y)$, then $\chi x = \chi y\,.$

If these axioms hold, then χ will be called a *precontraction*. If in addition,
(CS) χ is surjective,
then χ will be called a *contraction* and the group is a *contraction group*. Axioms (CA) and (CS) together show that every archimedean ordered contraction group must be trivial. Further, $(G,+,-,0,<,\chi)$ will be called *centripetal*, if it satisfies
(CP) $\forall x \in G\setminus\{0\}:\ |x| > |\chi x|\,,$
and it will be called *centrifugal*, if it satisfies
(CF) $\forall x \in G\setminus\{0\}:\ |x| < |\chi x|\,.$
In the sequel, we will write (G,χ) instead of $(G,+,-,0,<,\chi)$.

Axiom (CA) may be expressed by the following recursive axiom scheme:

$$\forall x,y:\ x \geq y > 0 \wedge ny \geq x \implies \chi x = \chi y \qquad (n\in\mathbb{N})\,.$$

Observe that axioms (C0), (C$-$) and (CA) together imply
(CZ) $\forall x \in G:\ \chi(zx) = \mathrm{sign}(z)\cdot\chi x \qquad (z\in\mathbb{Z})\,.$
Observe further that by axiom (C$\leq$), $x \leq y \leq z$ and $\chi x = \chi z$ implies $\chi y = \chi z$. This shows that in the presence of axiom (C$\leq$), the axiom scheme (CA) may be replaced by the single axiom
(CA$'$) $\forall x \in G:\ \chi(2x) = \chi x\,.$

This and all other axioms are immediately seen to be elementary in the language $\mathcal{L}_{\rm cg}$. All axioms are universal, except for the surjectivity axiom (CS). Since properties described by universal axioms are inherited by substructures, we have:

Lemma 3.1. *Every substructure S of a precontraction group (G,χ) is again a precontraction group. If (G,χ) is centripetal (resp. centrifugal), then so is S.*

We will need a further axiom scheme which is not universal. Namely, $(G,+,0)$ is *divisible* if it satisfies
(D) $\forall x\,\exists y:\ ny = x \qquad (0\neq n\in\mathbb{N})\,.$

We will consider the theory of divisible centripetal contraction groups. Some results also hold for the theory of divisible centrifugal contraction groups. We will frequently need that the groups are nontrivial, that is, they satisfy the axiom $\exists x:\ x\neq 0$. Before we continue, let us put together some technical preliminaries. Using the natural valuation v, we may express axiom (CA) in the following way:
(CV1) $\forall x,y:\ vx = vy \wedge \mathrm{sign}(x) = \mathrm{sign}(y) \Longrightarrow \chi x = \chi y$
(but note that v is neither a symbol in our language $\mathcal{L}_{\rm cg}$ nor definable in the theory of divisible centripetal or centrifugal contraction groups). Sometimes, we will only

be interested in equality up to the sign; instead of writing $|a| = |b|$ we will then write $a = \pm b$. Then, (CV1) reads as follows:

(CV2) $\forall x, y :\ vx = vy \Longrightarrow \chi x = \pm \chi y$.

From (V3), (NV3) and (V4), (NV4) we way infer:

(CV3) $\chi(\sum_{1\leq i\leq n} x_i) = \chi x_m \quad \text{if } vx_m < vx_i \text{ for all } i \neq m,$

(CV4) $\forall x, y :\ v(x-y) > vx \Rightarrow \chi x = \chi y$.

From (CV2) and (NV2) together with (C$\leq$), one may deduce

(CV5) $\forall x, y :\ vx \leq vy \Longrightarrow |\chi x| \geq |\chi y|$.

The following lemma collects some generalities about precontractions.

Lemma 3.2. *Let (G, χ) be a precontraction group. Then the following assertions hold:*

a) $\operatorname{sign}(a) = \operatorname{sign}(\chi a)$ *for every $a \in G$, hence $\chi G^{<0} \subset G^{<0}$ and $\chi G^{>0} \subset G^{>0}$. Moreover, $\chi G^{<0} = -\chi G^{>0}$.*

b) χ is centripetal if and only if $v(\chi a) > va$ for all $a \in G \setminus \{0\}$. Similarly, χ is centrifugal if and only if $v(\chi a) < va$ for all $a \in G \setminus \{0\}$.

c) Every nontrivial centripetal precontraction group is densely ordered. The same is true for every nontrivial centrifugal contraction group.

Proof: a): If $a < 0$ then by (C$\leq$) and (C0), $\chi a < \chi 0 = 0$. Similarly, $a > 0$ is shown to imply $\chi a > 0$. The last assertion follows directly from (C$-$).

b): Assume that χ is centripetal. Then for every $a \in G \setminus \{0\}$, we have $|\chi a| < |a|$ and thus, $v(\chi a) \geq va$. But for every $b \in G$ with $va = vb$ it follows from (CA) and (CP) that $|\chi a| = |\chi b| < |b|$. Hence, $v(\chi a) = va$ is impossible. The proof for centrifugal precontractions is similar.

c): A centripetal precontraction group cannot have a least positive element 1 since by (C0), (C$\leq$) and (CP), $0 < \chi g < g$ for every $g \in G^{>0}$. Now assume 1 to be the least positive element in a centrifugal contraction group (G, χ). Then there is some $a \in G$ such that $\chi a = 1$. But then by (C0), (C$\leq$) and (CF), $0 < a < 1$, a contradiction. □

Lemma 3.3. *Assume $(G, \chi) \subset (H, \chi)$ to be an extension of precontraction groups. Let $b \in H$ such that $vb \notin vG \cup \{\infty\}$ and $\chi b = a \in G$. Then $(G + \mathbb{Z}b, \chi)$ is a precontraction group with $\chi(G + \mathbb{Z}b) = \chi G \cup \{a, -a\} \subset G$. Moreover, the extension of χ from (G, χ) to $G + \mathbb{Z}b$ is uniquely determined by the assignment $\chi b = a$.*

Proof: Since $vb \notin vG$, (V3) shows that every element in $G + \mathbb{Z}b$ has value either in $vG \cup \{\infty\}$ or equal to vb. Let $d = g + nb \in G + \mathbb{Z}b$. Then $\chi d = \chi nb = \operatorname{sign}(n) \cdot a$ if $vd = vb$, and $\chi d = \chi g$ if $vd = vg \in vG \cup \{\infty\}$. Hence, the extension of χ from G to $G + \mathbb{Z}b$ is uniquely determined by $\chi b = a$. It also proves $\chi(G + \mathbb{Z}b) = \chi G \cup \{a, -a\} \subset G$. Consequently, $G + \mathbb{Z}b$ is closed under χ and thus a precontraction group by virtue of Lemma 3.1. □

Lemma 3.4. *Assume $(G, \chi) \subset (H, \chi)$ to be an extension of precontraction groups. Let $a \in G \setminus \chi G$ and $b \in H$ such that $a = \chi b$. Then $vb \notin vG \cup \{\infty\}$, and $(G + \mathbb{Z}b, \chi)$ is a precontraction group satisfying $\chi(G + \mathbb{Z}b) \subset G$. Moreover, if*

$(G,\chi)\subset(H',\chi')$ is a second extension of precontraction groups and if $b'\in H'$ such that $a=\chi'b'$, then the assignment $b\mapsto b'$ induces an isomorphism $G+\mathbb{Z}b\rightarrow G+\mathbb{Z}b'$ of precontraction groups over G.

Proof: If there would exist some $c\in G$ such that $vb=vc$ then in view of (CV2), $a=\chi b=\pm\chi c\in\chi G$. Hence, $a\notin\chi G$ implies $vb\notin vG\cup\{\infty\}$, and from Lemma 3.3 it follows that $(G+\mathbb{Z}b,\chi)$ is a precontraction group with $\chi(G+\mathbb{Z}b)\subset G$.

We show that the ordering on $vG\cup\{vb\}$ is uniquely determined by (G,χ) and the assignment $\chi b=a$. Assume w.l.o.g. that $a\in G^{<0}$. Then $b\in H^{<0}$ by part a) of Lemma 3.2. Let $c\in G^{<0}$. Since $vc\neq vb$, (NV2) shows that $vc<vb$ if and only if $c<b$ and $vc>vb$ if and only if $c>b$. Then by virtue of (C$\leq$), $vc<vb$ if and only if $\chi c<a$ and $vc>vb$ if and only if $\chi c>a$.

Now let also b' be as in the hypothesis. By what we have proved already, $vb\notin vG\cup\{\infty\}$ and $vb'\notin vG\cup\{\infty\}$, and the assignment $vb\mapsto vb'$ induces an order isomorphism over vG of the two sets $vG\cup\{vb\}$ and $vG\cup\{vb'\}$ (which are endowed with the restrictions of the ordering of vH resp. vH'). From Lemma 2.5 it follows that the groups $G+\mathbb{Z}b$ and $G+\mathbb{Z}b'$, endowed with the restrictions of the ordering of H resp. H', are isomorphic over G by sending b to b'. By the uniqueness statement of Lemma 3.3, this isomorphism is also an isomorphism of precontraction groups. □

There are very simple examples for precontraction groups. The map "sign" is itself a precontraction on $\mathbb{Z}$, but it is neither centrifugal nor centripetal. It satisfies $\forall x\in G:\ |x|\geq|\chi x|$, but this does not remain true in extensions of $\mathbb{Z}$ where 1 is not the least positive element. $\mathbb{Z}$ does not admit a centrifugal or centripetal precontraction since it is archimedean. Indeed, it follows from part b) of Lemma 3.2 that a nontrivial centrifugal or centripetal precontraction group must have infinite rank.

Consider the Hahn sums $\coprod_{\mathbb{N}}\mathbb{Z}$ and $\coprod_{-\mathbb{N}}\mathbb{Z}$ where $\mathbb{Z}$ stands for the ordered group $(\mathbb{Z},+,-,0,<)$, and $\mathbb{N}$ resp. $-\mathbb{N}$ stands for the positive resp. negative integers with their usual ordering. On the first Hahn sum, we may define a precontraction in the following way: if $(a_i)_{i\in\mathbb{N}}\in\coprod_{\mathbb{N}}\mathbb{Z}$ and if i_0 is the minimal index such that $a_{i_0}\neq 0$, then we set

$$\chi\,(a_i)_{i\in\mathbb{N}}=\mathrm{sign}(a_{i_0})\cdot e_{i_0+1}$$

where e_i denotes the element of $\coprod_{\mathbb{N}}\mathbb{Z}$ which has a 1 at the index i and zeros everywhere else. The resulting centripetal precontraction group will be denoted by $\mathcal{P}_{\mathrm{cp}}$. Analogously, if $(a_i)_{i\in-\mathbb{N}}\in\coprod_{-\mathbb{N}}\mathbb{Z}$ and if i_0 is the maximal index such that $a_{i_0}\neq 0$, then we set

$$\chi\,(a_i)_{i\in\mathbb{N}}=\mathrm{sign}(a_{i_0})\cdot e_{i_0-1}$$

where $e_i\in\coprod_{-\mathbb{N}}\mathbb{Z}$ is defined as above. The resulting centrifugal precontraction group will be denoted by $\mathcal{P}_{\mathrm{cf}}$. These examples are as representative as they can be:

Lemma 3.5. *$\mathcal{P}_{\mathrm{cp}}$ is the prime structure of the elementary class of nontrivial centripetal precontraction groups. Analogously, $\mathcal{P}_{\mathrm{cf}}$ is the prime structure of the elementary class of nontrivial centrifugal precontraction groups.*

Proof: We will show that every substructure S of a centripetal precontraction group generated by one element a is isomorphic to $\mathcal{P}_{\mathrm{cp}}$. Indeed, we infer from part b) of Lemma 3.2 that $v(\chi^n a)$, $n\in\mathbb{N}$, is a strictly increasing sequence in the value set vS. Let S_0 be the subgroup of S which is generated by all the elements $\chi^n a$. Every element s of S_0 is then a finite sum $\sum_{j=0}^{\infty} \nu_j \chi^j a$ with coefficients $\nu_j \in \mathbb{Z}$, only finitely many of them nonzero. In view of (V3) and (NV1), we find $vs = v(\chi^{j_0} a)$ if $j_0 = \min\{j \in \mathbb{N} \mid \nu_j \neq 0\}$. Then it follows from (CV2) that $\chi s = \pm \chi^{j_0+1} a$. This shows that S_0 is closed under χ and hence, $S_0 = S$.

Now set $h(\chi^n a) = e_n$ for $n \in \mathbb{N}$. By Lemma 2.5, this assignment extends linearly to an order and valuation preserving isomorphism from S onto $\coprod_{\mathbb{N}} \mathbb{Z}$.

The proof for the centrifugal case is similar. In this case, the sequence $v(\chi^n a)$, $n \in \mathbb{N}$, is strictly decreasing, and we have to set $j_0 = \max\{j \in \mathbb{N} \mid \nu_j \neq 0\}$. □

An extension $(G, v) \subset (H, v)$ of valued groups is called *rank preserving* $vH = vG$ (precisely speaking, if the induced embedding of vG in vH is onto). For example, the divisible hull $\tilde{G}$ of G is a rank preserving extension of (G, v) since every element in $\tilde{G}$ is archimedean equivalent to some element of G. But if $va = vb$, then every precontraction on H will satisfy $\chi a = \pm \chi b$ by virtue of (CV2). On the other hand, if χ is a precontraction on G, then it may be extended to H by setting $\chi a = \mathrm{sign}(a) \cdot \chi |b|$. This yields $\chi H = \chi G$, showing that H is closed under χ and thus, that (H, χ) is a precontraction group. Hence, we have:

Lemma 3.6. *Let (G, χ) be a precontraction group. Then for every extension $(H, <)$ of $(G, <)$ which is rank preserving with respect to the natural valuation, χ extends in a unique way to a precontraction χ on H (under preservation of the properties "centripetal" and "centrifugal"), and we have $\chi H = \chi G$. More precisely, if $(G, \chi) \subset (H', \chi')$ is an extension of precontraction groups such that $(H, <) \subset (H', <)$, then the restriction of χ' to H coincides with χ.*

In particular, these assertions hold for $H = \tilde{G}$.

If the rank preserving extension $G \subset H$ is nontrivial, then $\chi H = \chi G \neq H$. Hence:

Corollary 3.7. *Let $(G, \chi) \subset (H, \chi)$ be a proper rank preserving extension of precontraction groups. Then (H, χ) cannot be a contraction group. In particular, this holds for the divisible hull of a non-divisible precontraction group.*

To prove even more, we introduce a new map which is associated to χ. We define $\rho_\chi : vG \to G^{<0}$ as follows: if $\alpha = va \in vG$, $a \in G^{<0}$, then $\rho_\chi \alpha = \chi a$. This is well defined since by (CV1), the definition does not depend upon the choice of the negative element a of value α. We have $\rho_\chi(vG) = \chi G^{<0}$: if $b = \chi a \in \chi G$, then $b = \rho_\chi va$. Hence, ρ_χ is surjective if and only if χ is. Moreover, ρ_χ preserves $\leq$: on the one hand, χ preserves $\leq$ by (C$\leq$); on the other hand, $va < va'$ for $a, a' \in G^{<0}$ implies $a < a'$.

Theorem 3.8. *Let $(G, \chi) \subset (H, \chi)$ be a nontrivial extension of precontraction groups. If (H, χ) is a contraction group, then $vH \setminus vG$ is infinite and thus, $(G, v) \subset (H, v)$ is not rank preserving.*

Proof: Since (H, χ) is a contraction group, we have $\chi H = H$. By assumption, $G \subset H$ is a nontrivial extension, hence there is an element $b \in \chi H \setminus \chi G$. Since ρ_χ is surjective, there is some $\beta \in vH \setminus vG$ such that $\rho_\chi \beta = b$. The fact that $vH \setminus vG$ is nonempty implies that $H \setminus G$ is infinite. Thus, the same is true for $\chi H \setminus \chi G$, in view of $\chi H = H$ and $\chi G \subset G$. Again by the surjectivity of ρ_χ, it follows that $vH \setminus vG$ is infinite. □

We will now show that every precontraction group is embeddable in a divisible contraction group.

LEMMA 3.9. *Every (centripetal resp. centrifugal) precontraction group* (G,χ) *is embeddable in a divisible (centripetal resp. centrifugal) contraction group* (H,χ).

Proof: (H,χ) will be the union over a chain of precontraction groups (G_n,χ), $n\in\mathbb{N}$. Let G_1 be the Hahn product $\mathbf{H}_{vG}\mathbb{R}$. Then there is an order and valuation preserving embedding of G in G_1, and the extension $(G,v)\subset(G_1,v)$ is rank preserving. Hence, there is a unique extension of χ from G to G_1.

Having constructed (G_n,χ), let us show how to obtain (G_{n+1},χ). We set

$$\Gamma := vG_n \cup (G_n^{<0}\setminus \chi G_n^{<0}) \;.$$

To define an ordering on Γ, we extend the orderings which already exist on vG_n and on G_n. Hence, we only have to give the order relation between two elements $\beta\in vG_n$ and $a\in G_n^{<0}\setminus\chi G_n^{<0}$. We let $a<\beta$ if $a<\rho_\chi\beta$; otherwise, we let $a>\beta$. Now, we let G_{n+1} be the Hahn product $\mathbf{H}_\Gamma\mathbb{R}$. Then there is a natural order and valuation preserving embedding of G_n in G_{n+1} which is induced by the embedding $vG_n\subset\Gamma$. We identify G_n with its image in G_{n+1}. It remains to define the extension of χ. Let $b\in G_{n+1}^{<0}$. If $vb=vb'\in vG_n$ for some $b'\in G_n^{<0}$ then necessarily, in view of (CV1), $\chi b=\chi b'$. If $vb=a\in G_n^{<0}\setminus\chi G_n^{<0}$, then we set $\chi b=a\in G_n\subset G_{n+1}$. For $b\in G_{n+1}^{>0}$, we set $\chi b=-\chi(-b)$. In this way, every element of G_n becomes an element of the range of χ in G_{n+1}. Consequently, χ will be surjective on the union of the G_n. Since all other axioms are universal, we see that this union is a contraction group. □

Let us now consider the question whether there are "closures" of precontraction groups in contraction groups. Let $(G,\chi)\subset(G',\chi)$ be an extension of precontraction groups. We will call (G',χ) a *contraction hull* of (G,χ) if it is a contraction group and has the following universal property:

(CH) if $(G,\chi)\subset(H,\chi)$ is any extension of precontraction groups and (H,χ) is a contraction group, then there is an embedding of (G',χ) in (H,χ) over (G,χ).

Similarly, we will call (G',χ) a *divisible contraction hull* of (G,χ) if it is a divisible contraction group and has the following universal property:

(CHD) if $(G,\chi)\subset(H,\chi)$ is any extension of precontraction groups and (H,χ) is a divisible contraction group, then there is an embedding of (G',χ) in (H,χ) over (G,χ).

Note that if (G',χ) is a contraction hull of a divisible precontraction group (G,χ) but is not itself divisible, then its divisible hull is *not* a divisible contraction hull of (G,χ) (cf. Theorem 3.8).

LEMMA 3.10. *For every precontraction group* (G,χ) *there exists a contraction hull and a divisible contraction hull. Such a hull* (G',χ) *may be chosen such that* ρ_χ *induces an order preserving bijection* $vG'\setminus vG\to G'^{<0}\setminus\chi G$. *Moreover, we can assume that for every* $b\in G'$ *there is some* $n\in\mathbb{N}$ *such that* $\chi^n b\in G$.

Proof: By Lemma 3.9, we may embed (G,χ) in some divisible contraction group (H,χ). We will construct a subgroup of H as follows. For the construction of the contraction hull, we set $G_0=G$. For the construction of the divisible contraction hull, we let G_0 be the divisible hull of G and endow it with the unique extension of the precontraction χ (cf. Lemma 3.6). Note that $\chi G_0 = \chi G \subset G$. Assume that G_ν for some ordinal ν is already constructed such that ρ_χ induces a bijection $vG_\nu \setminus vG \to \chi G_\nu^{<0} \setminus \chi G$ (for $\nu = 0$, this bijection is empty). If χ is not surjective on G_ν and hence there is some $a \in G_\nu^{<0} \setminus \chi G_\nu$, then we choose $b \in H$ such that $\chi b = a$, and we set $G_{\nu+1} := G_\nu + \mathbb{Z}b$ (resp. $G_{\nu+1} := G_\nu + \mathbb{Q}b$ if we are constructing the divisible contraction hull). From Lemma 3.3 we know that $G_{\nu+1}$ together with the restriction of χ is a precontraction group having $vG_\nu \cup \{vb\}$ as its value set and satisfying $\chi G_{\nu+1} = \chi G_\nu \cup \{\pm a\} \subset G_\nu$. The map ρ_χ sends vb to $\chi b = a$ which is the only element in $\chi G_{\nu+1}^{<0} \setminus \chi G_\nu = \{a\}$. Hence, ρ_χ induces a bijection $vG_{\nu+1} \setminus vG \to \chi G_{\nu+1}^{<0} \setminus \chi G$. Since $\chi G_{\nu+1} \subset G_\nu$, $G_{\nu+1}$ inherits the property that for every element b there is some $n \in \mathbb{N}$ such that $\chi^n b \in G$.

If λ is a limit ordinal and if we have constructed G_ν for all $\nu < \lambda$, then we let G_λ be the union over all G_ν. Then G_λ is again a precontraction group (the theory of precontraction groups being universal). Still, ρ_χ induces a bijection $vG_\lambda \setminus vG \to \chi G_\lambda^{<0} \setminus \chi G$, and for every $b \in G_\lambda$ there is some $n \in \mathbb{N}$ such that $\chi^n b \in G$. Since at every step we are constructing a nontrivial extension but remain in H, this process is bounded by the successor cardinal κ^+ of the cardinality κ of H. Hence, we will arrive at some G_μ for an ordinal $\mu < \kappa^+$ where χ is surjective. That is, (G_μ, χ) is a contraction group (resp. a divisible contraction group). We set $G' := G_\mu$. Since now $G' = \chi G'$ holds, we have that ρ_χ induces a bijection $vG' \setminus vG \to G'^{<0} \setminus \chi G$ (which is order preserving since ρ_χ preserves $\leq$).

Now let $(G, \chi) \subset (H', \chi')$ be any extension of precontraction groups and assume that (H', χ') is a contraction group (resp. a divisible contraction group). Assume that we have already embedded (G_ν, χ) in (H', χ') over (G, χ); we may identify it with its image in (H', χ'). Now we have to show how this embedding extends to $(G_{\nu+1}, \chi)$. Let $a \in G_\nu$ and $b \in H$ be chosen as above. Choose $b' \in H'$ such that $\chi' b' = a$. Then Lemma 3.4 shows that the precontraction groups $(G_{\nu+1}, \chi)$ and $(G_\nu + \mathbb{Z}b', \chi')$ (resp. $(G_\nu + \mathbb{Q}b', \chi')$, using also Lemma 3.6) are isomorphic over (G_ν, χ). For a limit ordinal λ, the embeddings of the (G_ν, χ), $\nu < \lambda$, extend canonically to an embedding of (G_λ, χ). It follows that (G', χ) is embeddable in (H', χ') over (G, χ). □

Note that the universal property of the contraction hull is somewhat weak: the embedding is not necessarily unique. Indeed, we have seen in the proof above that there is some arbitrariness in the construction of the isomorphism, namely, the choice of $b' \in H'$ satisfying $\chi' b' = a$ is not canonical: we can take any $b'' \in H'$ with $vb'' = vb'$ instead. So we are not able to deduce the uniqueness of the contraction hull from the usual standard argument of category theory. However, uniqueness is not needed for our further results.

Applying the last lemma to the prime structures $\mathcal{P}_{\mathrm{cp}}$ and $\mathcal{P}_{\mathrm{cf}}$, we obtain:

Corollary 3.11. *The elementary classes of nontrivial centripetal (resp. centrifugal) contraction groups and of nontrivial divisible centripetal (resp. centrifugal) contraction groups have prime models.*

Further, the contraction hulls have a good model theoretic property:

Lemma 3.12. *Let $(G, \chi) \subset (G'', \chi)$ be an extension of precontraction groups and assume that (G, χ) is a contraction group (resp. a divisible contraction group).*

If (G,χ) is existentially closed in (G'',χ), then it is existentially closed in every contraction hull (resp. divisible contraction hull) (G',χ) of (G'',χ).

Proof: If (G,χ) is existentially closed in (G'',χ), then by Lemma 2.3, (G'',χ) is embeddable in every $|G''|^+$-saturated elementary extension $(G,\chi)^*$ of (G,χ). We identify (G'',χ) with its image in $(G,\chi)^*$. As an elementary extension, $(G,\chi)^*$ is a (divisible) contraction group like (G,χ). It follows from the universal property of (divisible) contraction hulls that every (divisible) contraction hull (G',χ) of (G'',χ) is embeddable in $(G,\chi)^*$ over (G'',χ). Again by Lemma 2.3, it now follows that (G,χ) is existentially closed in (G',χ). □

Let $(G,\chi)\subset(H,\chi)$ be an extension of precontraction groups. For a given $b\in H$, we will consider the cut

$$(\{g\mid G\ni g\leq b\}\,,\,\{g\mid G\ni g>b\}) \tag{1}$$

induced by b in G and its image under χ:

$$(\{\chi g\mid G\ni g\leq b\}\,,\,\{\chi g\mid G\ni g>b\})\,. \tag{2}$$

By virtue of (C$\leq$),

$$\{\chi g\mid G\ni g\leq b\}\ \leq \chi b\leq\ \{\chi g\mid G\ni g>b\}\,. \tag{3}$$

Now assume (G,χ) to be a contraction group. Then $\chi G=G$ and (2) is a quasicut in G which is realized by χb. If the sets of (2) have a nonempty intersection, then it consists of just one element which must be χb, implying that $\chi b\in G$. If the intersection of the two sets is empty, then (2) is a cut. In particular, this is the case if $\chi b\notin G$ and then, (2) is the cut induced by χb in G. If (2) is a cut which is not realized by any element of G, then $\chi b\notin G$ and the cut (1) determines uniquely the cut induced by χb in G. But if there is an element $a\in G$ which realizes (2), then the cut (1) may not determine whether $\chi b\in G$. Indeed, we then only know that there is no element of G between a and χb. This happens if $a=\chi b$, but it may also happen that H contains an element d such that $vd>vG$ and $\chi b=a+d$. Let us discuss the situation in general:

Lemma 3.13. *Let $(G,<)\subset(H,<)$ be an extension of ordered groups. Assume that two elements $b_1\in H$ and $b_2\in G$ realize the same quasicut in $(G,<)$. Then there is no element of G properly between b_1 and b_2; hence, there is no element of G properly between b_1-b_2 and 0. We have two cases:*

a) If b_1-b_2 is archimedean equivalent to some element of $G\setminus\{0\}$ (i.e., $v(b_1-b_2)\in vG$), then $v(b_1-b_2)$ is the maximal element in vG and $(G,<)$ is discretely ordered.

b) If b_1-b_2 is not archimedean equivalent to some element of $G\setminus\{0\}$, then it is archimedean smaller than all nonzero elements of G, that is, $v(b_1-b_2)>vG$. If $b_2\neq 0$, then it follows that $vb_1=vb_2$ and $\mathrm{sign}(b_1)=\mathrm{sign}(b_2)$.

In particular, if $(G,<)$ is dense and vG is cofinal in vH then necessarily, $b_1=b_2$.

Proof: Assume that there is no element of G properly between b_1-b_2 and 0. If there were $a\in G\setminus\{0\}$ such that $va>v(b_1-b_2)$ then a would be archimedean smaller than b_1-b_2 and thus, a or $-a$ would lie properly between b_1-b_2 and 0, contrary to our assumption.

Suppose that b_1-b_2 is archimedean equivalent to $a\in G\setminus\{0\}$, that is, $va=v(b_1-b_2)$. Then by what we have shown, it follows that va is the maximal element of vG. Consequently, the convex subgroup $\{x \in G \mid vx \geq va\}$ is isomorphic to $\{x \in G \mid vx \geq va\}/\{x \in G \mid vx > va\}$ and hence archimedean. It is an ordered subgroup of the archimedean group $\{x \in H \mid vx \geq va\}/\{x \in H \mid vx > va\}$ which contains the nonzero image of $b_1 - b_2$. By our assumption on $b_1 - b_2$ it follows that there is no element of $\{x \in G \mid vx \geq va\}$ properly between this image and zero. Since both groups are archimedean, it follows that $\{x \in G \mid vx \geq va\}$ must be discretely ordered, and the same is consequently true for G.

Now suppose that $b_1 - b_2$ is not archimedean equivalent to any element of $G\backslash\{0\}$. Then by our initial argument, $v(b_1 - b_2) > vG$. Since $b_2 \in G$ by assumption, this yields $v(b_1 - b_2) > vb_2$, showing that $vb_1 = vb_2$ and $\operatorname{sign}(b_1) = \operatorname{sign}(b_2)$ by virtue of (V4) and (NV4).

If $(G, <)$ is dense then it is not discretely ordered and the first case is impossible. If vG is cofinal in vH then the second case is impossible unless $v(b_1 - b_2) = \infty$, that is, $b_1 = b_2$. □

We will now apply this lemma to the quasicut (2).

Lemma 3.14. *Let $(G, \chi) \subset (H, \chi)$ be an extension of centripetal or centrifugal precontraction groups and assume that (G, χ) is a contraction group. Let $0 \neq b \in H$.*

a) Suppose that $a \in G \setminus \{0\}$ realizes (2). Then $v\chi b = va \in vG$ and thus, $\chi^2 b = \pm\chi a \in G$.

b) Suppose that vG is cofinal in vH and $a \in G \setminus \{0\}$ realizes (2). Then $\chi b = a$.

c) If (2) is not realized by any element of G, then $\chi b \notin G$ and the cut induced by χb in G is equal to (2) and thus uniquely determined by the cut (1). The same holds if 0 realizes (2).

Proof: Since we assume (G, χ) to be a centripetal or centrifugal contraction group, it is dense by part c) of Lemma 3.2, showing that part a) of the foregoing lemma cannot apply to (G, χ). Further, $0 \neq b \in H$ implies $\chi b \neq 0$. Hence, part a) and part b) of our present lemma are direct consequences of the foregoing lemma. The first assertion of c) was already discussed above.

Now assume that 0 realizes (2). Then the foregoing lemma shows that $v\chi b = v(\chi b - 0) > vG$. Since $\chi b \neq 0$, this yields $\chi b \notin G$. Consequently, (2) is the cut induced by χb in $(G, <)$. □

Lemma 3.15. *Let $(G, \chi) \subset (H, \chi)$ and $(G, \chi) \subset (H', \chi)$ be two extensions of centripetal or centrifugal contraction groups. Further, let $b \in H$ be such that $\chi^n b \notin G$ for all $n \in \mathbb{N}$. Suppose that some $b' \in H' \setminus \{0\}$ induces in G the same cut as b. Then also $\chi^n b' \notin G$ for all $n \in \mathbb{N}$, and both $(G + \sum_{n\in\mathbb{N}} \mathbb{Z}\chi^n b, \chi)$ and $(G + \sum_{n\in\mathbb{N}} \mathbb{Z}\chi^n b', \chi)$ are precontraction groups. Moreover, there is an isomorphism of these two groups over (G, χ), sending $\chi^n b$ to $\chi^n b'$ for all $n \in \mathbb{N}$.*

Proof: We show that the sets $G \cup \{\chi^n b \mid n \in \mathbb{N}\}$ and $G \cup \{\chi^n b' \mid n \in \mathbb{N}\}$, endowed with the restrictions of the orders of H resp. H', are order isomorphic over G by sending $\chi^n b$ to $\chi^n b'$ for every $n \in \mathbb{N}$, and that $v\chi^n b \notin vG$ for every $n \in \mathbb{N}$. By assumption, b and $b' \neq 0$ induce the same cut in G. Since $\chi^2 b \notin G$, we have $vb \notin vG$, and part a) of Lemma 3.14 shows that $b' \notin G$. We proceed by induction on $n \geq 0$: Suppose that we have already shown that $v\chi^n b \notin vG$ and $\chi^n b' \notin G$ and that

$$\mathrm{Cut}_n(b) := (\{\chi^n g \mid G \ni g \leq b\}, \{\chi^n g \mid G \ni g > b\})$$

is the cut induced by both $\chi^n b$ and $\chi^n b'$ in G. Since we know that $\chi^2(\chi^n b) = \chi^{n+2}b \notin G$, it follows that $v\chi^{n+1}b \notin vG$, and part a) of Lemma 3.14 shows that $\mathrm{Cut}_{n+1}(b)$ is not realized by any element of $G \setminus \{0\}$. We have $\chi^n b, \chi^n b' \neq 0$ since $\chi^n b, \chi^n b' \notin G$. Using part c) of Lemma 3.14, we may deduce that $\chi^{n+1}b' \notin G$ and that $\mathrm{Cut}_{n+1}(b)$ is the cut induced by both $\chi^{n+1}b$ and $\chi^{n+1}b'$ in G. This completes the induction step. It follows that for all $n \in \mathbb{N}$, the order relation between $\chi^n b$ and a given element $g \in G$ is the same as between $\chi^n b'$ and g. Furthermore, the ordering on the sets $\{\chi^n b \mid n \in \mathbb{N}\}$ and $\{\chi^n b' \mid n \in \mathbb{N}\}$ is given by our condition that both (H,χ) and (H',χ) be centripetal or centrifugal contraction groups. Hence, $\chi^n b \mapsto \chi^n b'$ induces an order isomorphism between the ordered sets $G \cup \{\chi^n b \mid n \in \mathbb{N}\}$ and $G \cup \{\chi^n b' \mid n \in \mathbb{N}\}$. By Lemma 2.5 it now follows that this isomorphism extends linearly to an order and valuation preserving isomorphism between the groups $G_b := G + \sum_{n\in\mathbb{N}} \mathbb{Z}\chi^n b$ and $G_{b'} := G + \sum_{n\in\mathbb{N}} \mathbb{Z}\chi^n b'$.

On the other hand, (V3) shows that every element in G_b has value either in $vG \cup \{\infty\}$ or equal to $v\chi^n b$ for some $n \in \mathbb{N}$. Consequently, the group G_b is closed under χ and thus, (G_b,χ) is a precontraction group. Moreover, it follows that the extension of χ from G to G_b is uniquely determined by the cut induced by b in G, together with the condition that $\chi^n b \notin G$ for all $n \in \mathbb{N}$. Since b' induces the same cut as b in G and $\chi^n b' \notin G$ for all $n \in \mathbb{N}$, it now follows that the above isomorphism between G_b and $G_{b'}$ also preserves the precontraction. □

Corollary 3.16. *Let $(G,\chi) \subset (H,\chi)$ be an extension of centripetal or centrifugal precontraction groups and assume that (G,χ) is a nontrivial divisible contraction group. Further, let $b \in H$ be such that $\chi^n b \notin G$ for all $n \in \mathbb{N}$. Then (G,χ) is existentially closed in the precontraction group*

$$(G_b,\chi) := (G + \sum_{n\in\mathbb{N}} \mathbb{Z}\chi^n b\,,\chi)\;.$$

Proof: Let $(G,\chi)^*$ be a $|G|^+$-saturated elementary extension of (G,χ). Since $(G,\chi)^*$ is divisible like (G,χ), Lemma 2.4 shows that there is some nonzero element b' of $(G,\chi)^*$ which induces the same cut in G as b. Hence, the previous lemma gives an embedding of (G_b,χ) in $(G,\chi)^*$ over (G,χ). Now it follows from Lemma 2.3 that (G,χ) is existentially closed in (G_b,χ). □

Corollary 3.17. *Let $(G,\chi) \subset (H,\chi)$ be an extension of nontrivial divisible centripetal contraction groups. Then there exists a divisible centripetal contraction group $(G',\chi) \subset (H,\chi)$ such that vG' is cofinal in vH and (G,χ) is existentially closed in (G',χ).*

Proof: If vG is not cofinal in vH then choose $b \in H \setminus \{0\}$ such that $vb > vG$. Since (H,χ) is centripetal, we have $v\chi^n b > vG$ for all $n \in \mathbb{N}$. By the foregoing lemma, (G,χ) is existentially closed in the precontraction group (G_b,χ). By Lemma 3.12, (G,χ) is also existentially closed in every divisible contraction hull of that group. By its universal property, such a divisible contraction hull can be chosen in (H,χ). If the value set of this new group is still not cofinal in vH, we may proceed by (possibly transfinite) induction (bounded by the cardinality of vH) to construct (G',χ) such that vG' is cofinal in vH. □

For centripetal contraction groups, we will now generalize the principle that we have used in the proof of Lemma 3.15. The elements b, b' appearing in that lemma seem to be "transcendental" in some sense. For the centripetal case, we will give a corresponding definition.

In the sequel, all precontraction groups are assumed to be centripetal.

Let $(G,\chi) \subset (H,\chi)$ be an extension of precontraction groups, G divisible, and (H',χ) a substructure of (H,χ) generated over G by one element b. Then as an abelian group, H' is not necessarily generated over G by only the element b. We form a sequence of generators for H' as an abelian group over G in the following way. If $G+\mathbb{Z}b$ is a rank preserving extension of G, then by Lemma 3.6, we have $\chi(G+\mathbb{Z}b) \subset G$. In this case, we find $H' = G+\mathbb{Z}b$, and our sequence only consists of the element $b_1 := b$; we set $g_1 := 0$ for later use. In this case, g_1, b_1 and vb_1 are uniquely determined by b. If on the other hand, the extension is not rank preserving, then there is an element g_1 in the divisible group G such that $v(b-g_1) \notin vG \cup \{\infty\}$. We set $b_1 := b - g_1$ and find that $G+\mathbb{Z}b = G+\mathbb{Z}b_1$. Note that vb_1 is uniquely determined by b, by virtue of Lemma 2.6. In view of (V3), every element of $G+\mathbb{Z}b_1$ has value either in $vG\cup\{\infty\}$ or equal to vb_1. Hence by (CV1), $\chi(G+\mathbb{Z}b_1) \subset G+\mathbb{Z}\chi b_1 \subset G+\mathbb{Z}b_1+\mathbb{Z}\chi b_1$. Having constructed b_n and the group $G+\sum_{i=1}^n \mathbb{Z}b_i$ such that $vb_1,\ldots,vb_n$ is a strictly increasing sequence of values not in $vG\cup\{\infty\}$ and uniquely determined by b, and such that $\chi(G+\sum_{i=1}^n \mathbb{Z}b_i) \subset G+\sum_{i=1}^n \mathbb{Z}b_i + \mathbb{Z}\chi b_n$, we proceed as follows.

Since χ is centripetal, we have $v\chi b_n > vb_n \geq \{vb_1,\ldots,vb_n\}$. By virtue of (V3), $\chi b_n \in G+\sum_{i=1}^n \mathbb{Z}b_i$ will thus imply $\chi b_n \in G$. If this is the case, then $G+\sum_{i=1}^n \mathbb{Z}b_i$ is closed under χ, and we let our sequence end with b_n.

If $G+\mathbb{Z}\chi b_n$ is a nontrivial rank preserving extension of G then we set $b_{n+1} := \chi b_n$ and $g_{n+1} := 0$. Then $vb_{n+1} = v\chi b_n > vb_n$. Note that in this case again, g_{n+1}, b_{n+1} and vb_{n+1} are uniquely determined by χb_n. This in turn is, up to the sign, uniquely determined by vb_n. Hence, vb_{n+1} is uniquely determined by vb_n and thus by b. The nonzero elements $b_1,\ldots,b_n$ have different values $\notin vG = v(G+\mathbb{Z}b_{n+1})$ which shows that every element d of $G+\sum_{i=1}^{n+1}\mathbb{Z}b_i$ has value either in $vG\cup\{\infty\}$ or equal to vb_i for some $i \leq n$. Hence by virtue of (CV3), $\chi(G+\sum_{i=1}^{n+1}\mathbb{Z}b_i) = \chi(G+\sum_{i=1}^{n}\mathbb{Z}b_i) \subset G+\sum_{i=1}^{n+1}\mathbb{Z}b_i$. Consequently, the group $G+\sum_{i=1}^{n+1}\mathbb{Z}b_i$ is closed under χ, and we let our sequence end with b_{n+1}.

If the extension is not rank preserving, then there is an element $g_{n+1}\in G$ such that $v(\chi b_n - g_{n+1}) \notin vG\cup\{\infty\}$ and we set $b_{n+1} := \chi b_n - g_{n+1}$. We find that

$$vb_{n+1} = v(\chi b_n - g_{n+1}) \geq v\chi b_n > vb_n$$

since otherwise, $v(\chi b_n - g_{n+1}) = vg_{n+1} \in vG\cup\{\infty\}$ by (V3). (Note that it is precisely the $\geq$-sign in the foregoing inequality that makes it impossible to construct analogous sequences with strictly decreasing values in centrifugal precontraction groups.) Again by Lemma 2.6, vb_{n+1} is uniquely determined by χb_n; as before, it follows that vb_{n+1} is uniquely determined by b. All elements $b_1,\ldots,b_{n+1}$ have different values $\notin vG\cup\{\infty\}$, so every element d of $G+\sum_{i=1}^{n+1}\mathbb{Z}b_i$ has value either in $vG\cup\{\infty\}$ or equal to vb_i for some $i\leq n+1$. If $vd = vb_{n+1}$ then $\chi d = \pm\chi b_{n+1}$ by (CV2). Otherwise, $\chi d \in \chi(G+\sum_{i=1}^n \mathbb{Z}b_i) \subset G+\sum_{i=1}^n \mathbb{Z}b_i + \mathbb{Z}\chi b_n = G+\sum_{i=1}^{n+1}\mathbb{Z}b_i$. Hence, $\chi(G+\sum_{i=1}^{n+1}\mathbb{Z}b_i) \subset G+\sum_{i=1}^{n+1}\mathbb{Z}b_i+\mathbb{Z}\chi b_{n+1}$. Since $vb_{n+1}\notin vG\cup\{\infty\}$, the procedure will again be repeated.

The sequence of elements b_i constructed in this way may be finite or infinite. If it is finite, the element b will be called *χ-algebraic over* (G,χ), and otherwise, b will be called *χ-transcendental over* (G,χ). These notions are welldefined, although the *shift elements* g_i and the sequence members b_i may not be uniquely determined. Indeed, in our construction, all values vb_i were uniquely determined by b, which also yields that all sequences constructed from b must have the same length. Every such sequence will be called a *characteristic sequence* of b over (G,χ). If $b_m \neq 0$ and $vb_m > vG$ for some m, then the sequence is infinite and $vb_i > vG$ holds for all $i \geq m$; the sequence beginning with b_m will then be called a *supersequence*. If vG is cofinal in vH, then there are no supersequences in H over (G,χ).

Note that "χ-algebraic" does not mean "algebraic" in the model theoretic sense. We will show in [K2] that only the elements of the divisible hull $\tilde{G}$ are algebraic over a precontraction group (G,χ).

By construction, the abelian group generated over G by all the elements of a characteristic sequence, is closed under χ and thus a precontraction group. On the other hand, all elements b_i are contained in H' which was the substructure of (H,χ) generated by b. We find that the elements of a characteristic sequence form a set of generators of H' as an abelian group over G. By construction, it is even a minimal set of generators.

For arbitrary b_n in a characteristic sequence, all inclusions in the chain $G \subset G+\mathbb{Z}b_n \subset G+\mathbb{Z}b_n+\mathbb{Z}b_{n-1} \subset \ldots \subset G+\sum_{i=1}^n \mathbb{Z}b_i$ are proper. Indeed, from our construction it follows that $vb_{j-1} < \{vb_j,\ldots,vb_n\}$ and there is thus no element of value vb_{j-1} in the group $G+\sum_{i=j}^n \mathbb{Z}b_i$, for every $j \leq n$. Since $\chi(G+\sum_{i=j}^n \mathbb{Z}b_i) \subset G+\sum_{i=j+1}^n \mathbb{Z}b_i + \mathbb{Z}\chi b_n$, we also see that

$$b_j + g \;\notin\; \chi(G+\sum_{i=j}^n \mathbb{Z}b_i) \quad \text{for every } j \leq n \text{ and all } g \in G.$$

Hence if $j \geq 2$, then

$$\chi b_{j-1} \;=\; b_j + g_{j-1} \;\notin\; \chi(G+\sum_{i=j}^n \mathbb{Z}b_i)\,. \tag{4}$$

Note that our assertions also hold for $\mathbb{Q}$ in the place of $\mathbb{Z}$. If $b_1,\ldots,b_n$ is the characteristic sequence of some χ-algebraic element b over (G,χ), then all groups in this chain are precontraction groups since for every $i \leq n$, the sequence $b_i,\ldots,b_n$ is a characteristic sequence of b_i. The construction carried through in the proof of Lemma 3.10 may be applied to $(G+\mathbb{Z}b_n,\chi)$ in the place of (G,χ). By virtue of (4), we may take the first adjoined elements to be $b_{n-1},\ldots,b_1$ successively. In this way, we obtain a contraction hull of $(G+\mathbb{Z}b_n,\chi)$ in (H,χ) which contains all elements of the given characteristic sequence of b. Hence, it also contains the precontraction group generated by b over (G,χ). If H is divisible, then we may replace $\mathbb{Z}$ by $\mathbb{Q}$ to obtain a divisible contraction hull. We have proved:

Lemma 3.18. *Let $(G,\chi) \subset (H,\chi)$ be an extension of centripetal precontraction groups and assume that G is divisible and (H,χ) is a contraction group (resp. a divisible contraction group). Let $b \in H$ be χ-algebraic over (G,χ) with characteristic sequence $b_1,\ldots,b_n$. Then there exists a contraction hull (resp. a divisible contraction hull) of $(G+\mathbb{Z}b_n,\chi)$ in (H,χ) which contains the precontraction group generated by b over (G,χ).*

In the sequel, let us assume (G,χ) to be a divisible centripetal contraction group. Let $b\notin G$. We will now examine the question whether the cuts induced in G by the elements of characteristic sequences of b are already determined by the cut

$$\mathrm{Cut}(b):=(\{g\mid G\ni g<b\}\,,\,\{g\mid G\ni g>b\})$$

induced by b in G. To this end, we need the notion of a v-cut that we have introduced in section 2. Bearing in mind the proof of Lemma 3.15 where we defined an isomorphism by an assignment $\chi^i b\mapsto\chi^i b'$, we will treat the following problem. Assume that $(G,\chi)\subset(H',\chi)$ is a second extension of centripetal precontraction groups and that $b'\in H'$ also induces the cut $\mathrm{Cut}(b)$ in G. Is it then possible to construct characteristic sequences of b and b' of equal length and having the same shift elements g_i for every i?

Let us fix a characteristic sequence $b_1,\dots$ of b over (G,χ). For every b_i in this sequence, we have

$$\mathrm{Cut}(b_i)=(\{g\mid G\ni g<b_i\}\,,\,\{g\mid G\ni g>b_i\})$$

since by definition, $b_i\notin G$.

Assume that b_m, $m\geq 1$, is an element of our characteristic sequence and that we have already shown $\mathrm{Cut}(b_{m-1})$ to depend only on $\mathrm{Cut}(b)$ (and not on the chosen characteristic sequence or the element $b\notin G$ inducing $\mathrm{Cut}(b)$). For $m=1$, this assumption is trivially true. If $m>1$, then assume in addition that we have already constructed the first elements $b'_1,\dots b'_{m-1}$ of a characteristic sequence of b', satisfying $b'_1=b'-g_1$ and $b'_i=\chi b_{i-1}-g_i$ as well as $\mathrm{Cut}(b_i)=\mathrm{Cut}(b'_i)$ for all $1<i<m$. For the case $m=1$, we set $C_1:=\mathrm{Cut}(b)=\mathrm{Cut}(b')$. For $m>1$, we set

$$C_m:=(\{\chi g\mid G\ni g<b_{m-1}\}\,,\,\{\chi g\mid G\ni g>b_{m-1}\})\;. \tag{5}$$

Note that a priori, we only know that (5) is a quasicut. But since b_m is an element of our characteristic sequence, $b_m+g_m\notin G$. As $\chi b_{m-1}=b_m+g_m$ realizes (5), this shows that (5) is a cut. Since by induction hypothesis, $\mathrm{Cut}(b_{m-1})=\mathrm{Cut}(b'_{m-1})$ depends only on $\mathrm{Cut}(b)$, also C_m depends only on $\mathrm{Cut}(b)$, and by virtue of (C$\leq$), both χb_{m-1} and $\chi b'_{m-1}$ realize C_m. In the sequel, let us treat the cases $m=1$ and $m>1$ simultaneously by taking the undefined expression "χb_0" to stand for "b".

Assume that C_m is not realized in G. There are two possibilities for C_m. If C_m is not a shifted v-cut, then by Corollary 2.2, there is no $g\in G$ such that $v(\chi b_{m-1}-g)\notin vG\cup\{\infty\}$ or $v(\chi b'_{m-1}-g)\notin vG\cup\{\infty\}$. Then necessarily, $g_m=0$ and $b_m=\chi b_{m-1}$ as well as $b'_m=\chi b'_{m-1}$ by our construction of characteristic sequences, implying also that $\mathrm{Cut}(b_m)=C_m=\mathrm{Cut}(b'_m)$. Moreover, b_m and b'_m are the last elements of the respective characteristic sequences.

If on the other hand, C_m is a shifted v-cut, then by Corollary 2.2, there is some $g\in G$ such that $v(\chi b_{m-1}-g)\notin vG\cup\{\infty\}$. Then by our construction, $vb_m=v(\chi b_{m-1}-g_m)\notin vG\cup\{\infty\}$ which by Corollary 2.2 yields that C_m-g_m is the unique v-cut among all shifts of C_m. By part b) of Lemma 2.1, we find that $v(\chi b'_{m-1}-g_m)\notin vG\cup\{\infty\}$, so we may take $b'_m:=\chi b'_{m-1}-g_m$. But also any other elements $\tilde b_m$, $\tilde b'_m$ continuing the characteristic sequences would have to satisfy $v\tilde b_m$, $v\tilde b'_m\notin vG\cup\{\infty\}$ and would thus induce v-cuts, which are then equal to C_m-g_m because of its uniqueness. Consequently, $\mathrm{Cut}(b_m)=C_m-g_m=\mathrm{Cut}(b'_m)$ only depends on $\mathrm{Cut}(b)$.

Now assume that C_m *is* realized by some $a\in G$. Then in view of Lemma 3.13, of which only part b) can apply here since G is dense, our construction of characteristic sequences yields $vb_m > vG$, and b_m is the beginning of a supersequence. Then b is χ-transcendental and for all $i \geq m$, we have $vb_i > vG$ and all cuts $\mathrm{Cut}_i(b)$ are the same and equal to one of the two possible cuts in G which are realized by 0 (these are $(G^{<0}, \{0\} \cup G^{>0})$ and $(G^{<0} \cup \{0\}, G^{>0})$). But the following pathological case may appear: we cannot exclude the possibility that $\chi b'_{m-1} = a$. In this case, the characteristic sequence of b' would end with b'_{m-1} and b' would be χ-algebraic. Then certainly, the assignment $b \mapsto b'$ would not extend to an isomorphism of precontraction groups over (G, χ). On the other hand, if our characteristic sequence of b does not eventually run into a supersequence, then for none of its elements b_i, the cut $\mathrm{Cut}(b_i)$ is realized in G. In this case, our procedure of constructing b'_i may be accomplished through the full length of the characteristic sequence of b, showing that the characteristic sequences of b' are at least as long as those of b. If symmetrically, we know that also the characteristic sequences of b' do not run into supersequences, then we may conclude that the characteristic sequences of b and b' are of equal length.

If the characteristic sequences of $b \in H$ over (G, χ) are infinite but do not run into supersequences, then b will be called *bounded χ-transcendental over* (G, χ). If vG is cofinal in vH, then there are no supersequences in H over (G, χ), and every χ-transcendental element is already bounded χ-transcendental.

Assume that $b \in H$ is bounded χ-transcendental over (G, χ) and that b' is as above, satisfying $\mathrm{Cut}(b) = \mathrm{Cut}(b')$. Then our construction yields an infinite characteristic sequence $b'_1, \ldots$ of b' with the same shift elements g_i and such that the cuts $\mathrm{Cut}(b_i) = \mathrm{Cut}(b'_i)$ are not realized in G, for all i. In particular, also b' is bounded χ-transcendental over (G, χ). Moreover, we know that all b_i and b'_i have values not contained in $vG \cup \{\infty\}$. Let us further note that every isomorphism ι between the ordered groups $(G + \sum_{i=1}^{\infty} \mathbb{Z} b_i, <)$ and $(G + \sum_{i=1}^{\infty} \mathbb{Z} b'_i, <)$ sending b_i to b'_i, will automatically respect the precontraction. Indeed, since every element in the first group has value either in $vG \cup \{\infty\}$ or equal to some vb_i, we only have to check whether $\iota\chi b_i = \chi b'_i$. But this is immediate: $\iota\chi b_i = \iota(b_{i+1} + g_{i+1}) = b'_{i+1} + g_{i+1} = \chi b'_i$.

Now the proofs of the following lemma and corollary are analogous to those of Lemma 3.15 and Corollary 3.16.

Lemma 3.19. *Let $(G, \chi) \subset (H, \chi)$ and $(G, \chi) \subset (H', \chi)$ be two extensions of centripetal contraction groups and assume that G is divisible. Further, let $b \in H$ be bounded χ-transcendental over (G, χ). Suppose that some $b' \in H'$ induces in G the same cut as b. Then also b' is bounded χ-transcendental over (G, χ), and there is an isomorphism of the precontraction groups generated by b resp. b' over (G, χ), sending b to b'.*

Corollary 3.20. *Let $(G, \chi) \subset (H, \chi)$ be an extension of centripetal precontraction groups and assume that (G, χ) is a nontrivial divisible contraction group. If (G_b, χ) is the precontraction group generated over G by one bounded χ-transcendental element $b \in H$, then (G, χ) is existentially closed in (G_b, χ).*

Next, we treat the case of extensions generated by one χ-algebraic element.

Lemma 3.21. *Let $(G, \chi) \subset (H, \chi)$ be an extension of centripetal precontraction groups and assume that (G, χ) is a nontrivial divisible contraction group. If (H', χ) is a substructure of (H, χ) generated over G by one χ-algebraic element b, then (G, χ) is existentially closed in (H', χ).*

Proof: We may assume $b \notin G$ since otherwise, the assertion is trivial. Pick a characteristic sequence $b_1, \ldots, b_n$ of b over (G, χ). Then the group $G + \mathbb{Z}b_n$ is a precontraction group, generated over (G, χ) by b_n. According to Lemma 3.18, there is a divisible contraction hull (H_0, χ) of $G + \mathbb{Z}b_n$ in (H, χ) which contains the precontraction group (G_b, χ) generated by b over (G, χ). If we are able to show that (G, χ) is existentially closed in $(G + \mathbb{Z}b_n, \chi)$, then it will follow from Lemma 3.12 that (G, χ) is also existentially closed in (H_0, χ) and thus also in (G_b, χ). This shows: w.l.o.g. we may assume from the start that $n = 1$ and $b_1 = b \notin G$. Then the cut induced by b in G is not realized in G, since otherwise, b_1 would be the beginning of a supersequence and b could not be χ-algebraic over (G, χ). We also know that $\chi b \in G$. Since (G, χ) is a contraction group, we may pick some $c \in G$ such that $\chi c = \chi b$. Let us assume that $c < b$; for $c > b$ the proof is analogous. Consider the following set of assertions:

$$\{\text{“}g < x \wedge \chi g = \chi x\text{”} \mid g \in G \wedge c \leq g < b\} \cup \{\text{“}g > x\text{”} \mid g \in G \wedge g > b\}\,. \tag{6}$$

This set is finitely satisfiable in (G, χ). For this, we only have to show that for every finite subset $\mathcal{F}$ of (6), there is some $x \in G$ for which all assertions of $\mathcal{F}$ are true. We write $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ with $\mathcal{F}_1$ a subset of the first and $\mathcal{F}_2$ a subset of the second set in (6). Let g_1 be the maximal element appearing in the assertions of $\mathcal{F}_1$. Then there is some element $g' \in G$ such that $g_1 < g' < b$ since otherwise, g_1 would realize the cut induced by b in G. Since $c \leq g_1 < g' < b$ and $\chi c = \chi b$, it follows from (C$\leq$) that $\chi g = \chi b = \chi g'$ for all g appearing in the assertions of $\mathcal{F}_1$, and for these, we also have $g \leq g_1 < g'$. On the other hand, for every $g > b$ we have $g > g'$. This shows that $x = g'$ satisfies all assertions in $\mathcal{F}$. Thus we have shown that the set (6) is finitely satisfiable in (G, χ). Hence, it is satisfiable in every $|G|^+$-saturated elementary extension $(G, \chi)^*$ of (G, χ). That is, there is some element b' in $(G, \chi)^*$ which induces in G the same cut as b and which satisfies $\chi b' = \chi b$. This yields that the assignment $b \mapsto b'$ defines an order preserving isomorphism from $G + \mathbb{Z}b$ onto $G + \mathbb{Z}b'$ over G. We have to show that this isomorphism is also an isomorphism of contraction groups. If $G \subset G + \mathbb{Z}b$ is not rank preserving, then by our choice of b (to be the first element of some characteristic sequence of b) we know that $vb \notin vG \cup \{\infty\}$, and our assertion follows from the uniqueness assertion of Lemma 3.3. If $G \subset G + \mathbb{Z}b$ is rank preserving, then the uniqueness assertion of Lemma 3.6 shows that every order preserving isomorphism from $G + \mathbb{Z}b$ onto $G + \mathbb{Z}b'$ over G will automatically be an isomorphism of contraction groups.

We have shown that (G_b, χ) embeds over (G, χ) in every $|G|^+$-saturated elementary extension of (G, χ). Hence, (G, χ) is existentially closed in (G_b, χ), as contended. $\square$

Now we are able to prove our main lemma.

Lemma 3.22. *Let $(G, \chi) \subset (H, \chi)$ be an extension of centripetal precontraction groups and assume that (G, χ) is a nontrivial divisible contraction group. Then (G, χ) is existentially closed in (H, χ).*

Proof: In view of Lemma 3.9, we may assume w.l.o.g. that (H, χ) is a divisible contraction group. By Corollary 3.17, we know that there exists a divisible contraction group (G', χ) in (H, χ) such that vG' is cofinal in vH and that (G, χ) is existentially closed in (G', χ). Now, it remains to show that (G', χ) is existentially closed in (H, χ). Hence, we may assume from the start that vG is cofinal in vH.

It suffices to show that (G,χ) is existentially closed in every substructure of (H,χ) which is finitely generated over G. Again in view of Lemma 3.9, we may thus assume w.l.o.g. that (H,χ) is a divisible contraction hull of a precontraction group which is finitely generated over (G,χ). Picking one of the generators out of a minimal set of, say, m generators, we may consider the substructure (G_1,χ) of (H,χ) generated by this element. We know that (G_1,χ) is a precontraction group. If we are able to show that (G,χ) is existentially closed in (G_1,χ), then by Lemma 3.10 and Lemma 3.12 there is a divisible contraction hull (G_1',χ) of (G_1,χ) in (H,χ) such that (G,χ) is existentially closed in (G_1',χ). Now (H,χ) is a divisible contraction hull of a precontraction group which is generated over (G_1',χ) by $m-1$ generators. This shows that we may proceed by induction on the number of generators. That is, we only have to show that (G,χ) is existentially closed in every substructure of (H,χ) which is generated over G by one element b. If b is χ-algebraic over (G,χ), then this is the assertion of Lemma 3.21. If b is χ-transcendental over (G,χ), then it is bounded χ-transcendental since vG is cofinal in vH, and an application of Lemma 3.20 now completes our proof. □

By Robinson's Test, the foregoing lemma implies:

THEOREM 3.23. *The theory of nontrivial divisible centripetal contraction groups is model complete.*

Now assume that (G,χ) is a common substructure of the two divisible centripetal contraction groups (H,χ) and (H',χ). By Lemma 3.1, (G,χ) is a centripetal precontraction group. By Lemma 3.10, there is a divisible contraction hull (G',χ) of (G,χ) in (H,χ) which embeds in (H',χ) over (G,χ). Let us identify (G',χ) with its image in (H',χ). By Lemma 3.5, the theory of centripetal contraction groups admits $\mathcal{P}_{\rm cp}$ as its prime structure. Hence if G is the trivial group, then we may replace (G,χ) by $\mathcal{P}_{\rm cp}$ to obtain that (G',χ) is nontrivial. Now (G',χ) is a nontrivial divisible centripetal contraction group, so from the model completenes stated in the preceding theorem we may infer that (H,χ) and (H',χ) are equivalent over (G',χ) and hence also over (G,χ). We have thus shown that the theory of divisible centripetal contraction groups is substructure complete, that is,

THEOREM 3.24. *The theory of nontrivial divisible centripetal contraction groups admits elimination of quantifiers.*

Combining this result with Lemma 3.9, we obtain:

THEOREM 3.25. *The theory of nontrivial divisible centripetal contraction groups is the model completion of the theory of centripetal precontraction groups.*

Since every two centripetal contraction groups contain the trivial group as a common substructure, substructure completeness yields

THEOREM 3.26. *The theory of nontrivial divisible centripetal contraction groups is complete.*

Since the axiom system {(OAG), (C0), (CS), (C$\leq$), (C$-$), (CA$'$), (CP), (D), $\exists x : x \neq 0$} is recursive, this theorem implies

THEOREM 3.27. *The theory of nontrivial divisible centripetal contraction groups is decidable.*

4. Other concepts of contraction groups

Although we have constantly used the natural valuation in the last sections, we have not put a symbol for it into the language of contraction groups. So the question arises whether we obtain similar results if we do. This would mean that we add a symbol for a binary relation which is interpreted by "$vx < vy$" or "$vx \leq vy$". Then certainly, to obtain completeness, there have to be axioms telling us about the relation between the valuation and the contraction. Note that the natural valuation of an ordered group is not elementarily axiomatizable. It is elementarily axiomatizable that a valuation be compatible with the ordering, but then there will always be models whose valuation is a proper coarsening of the natural valuation. The most direct way to fix the relation between valuation and contraction is to say that the valuation is the *valuation associated to* χ:

(VC) $\forall x, y \in G: \; vx = vy \;\Leftrightarrow\; \chi x = \pm \chi y\,.$

This valuation is always a coarsening of the natural valuation; it coincides with the natural valuation for instance on the prime structure and prime models that we have constructed. Since it is definable in the theory of precontraction groups, the definition just being axiom (VC), all model theoretic results remain true for this expansion.

It is also possible to axiomatize elementarily that for every nonzero element a the set $\{vb \mid \chi b = a\}$ contains precisely n values, where $n > 0$ is a fixed natural number or ∞. It is rather likely that the model theoretic results also carry over to these expansions.

Let us now consider valued abelian groups which are not ordered. In this case, an adequate axiom system for contraction groups might be (OAG), (C0), (CS) together with

(CV′) $\forall x, y \in G: \; vx = vy \;\Rightarrow\; \chi x = \chi y\,,$

and in view of part b) of Lemma 3.2, we would express the property "centripetal" by the axiom $\forall x \in G: \; x \neq 0 \;\Rightarrow\; v\chi x > vx$, and the property "centrifugal" by the axiom $\forall x \in G: \; x \neq 0 \;\Rightarrow\; v\chi x < vx$.

Open Problem 1: Determine the algebraic and model theoretic properties of this theory.

Note that the contraction transports (a coarsening of) the ordering of the value set into the group. But this ordering will not be compatible with the addition. We do not know whether this is a sound situation. If so, then there arises a further question:

Open Problem 2: Do there exist o-minimal or weakly o-minimal expansions or models of this theory?

Finally, on abelian groups which are not ordered or valued, there may still be classes of elements which are appropriate to be contracted. For example, if the group is an R-module for some ring R, we may consider the axiom scheme

(CR) $\forall x \in G: \; rx \neq 0 \;\Rightarrow\; \chi rx = \chi x \qquad (r \in R)\,,$

that is, for every element a in the group, the set χRa is a singleton. Combined again with the surjectivity, this describes rather "big" modules.

Open Problem 3: Study (CR) or similar axioms in combination with the surjectivity (CS).

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