Open Problems in connection with additive polynomials

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A. The main problems.

Open Problem 1 Is the elementary theory of the valued field $\mathbb{F}_p((t))$ model complete? Does it admit quantifier elimination in some natural language? Is it decidable? If yes, what would be a complete recursive axiomatization? What about other non-perfect henselian defectless valued fields of positive characteristic?

Note: The axiom system "Henselian defectless valued field of characteristic p with residue field $\mathbb{F}_p$ and value group a $\mathbb{Z}$-group" is not complete.

Open Problem 2 Does every place P of an algebraic function field $F|K$ of positive characteristic admit local uniformization, provided that the residue field extension $FP|K$ is separable?

Note: YES for Abhyankar places. YES for all places after a finite extension of the function field ("alteration"), with valuation theoretical proof. Both results have analogues for the arithmetic case.

B. Purely wild extensions and tame fields.

Open Problem 3 What is the structure of finite purely wild extensions that are not minimal? What can we say about their minimal polynomials?

Open Problem 4 In which way can we build information about purely wild extensions into structures (like angular component maps or AMC structures, cf. [Ku2]) which help to eliminate quantifiers for valued fields (like henselian fields with residue characteristic 0, p-adically closed fields, algebraically maximal Kaplansky fields)? Which structure would allow us to eliminate quantifiers for tame fields?

C. The optimal approximation property (OAP).

Open Problem 5 The OAP of images of additive polynomials is independent of the above stated axiom system for $\mathbb{F}_p((t))$. Do we obtain a complete axiom system if we add it?

Open Problem 6 Do all images of additive polynomials on maximal fields have the OAP? How do we construct valued fields which have the OAP for all images of additive polynomials? How do we get from one field which violates this property to a "next larger" one that has the property?

Open Problem 7 Which other definable subsets of valued fields have the OAP?

Open Problem 8 Is there a "small" set of "representative" additive polynomials such that the OAP of their images implies the OAP for the images of all additive polynomials?

D. $K[\varphi]$-modules.

Open Problem 9 What do Rohwer's results (see [Roh]) tell us about the model theory of the valued field $\mathbb{F}_p((t))$?

Open Problem 10 Does the elementary theory of $\mathbb{F}_p((t))$ as an $\mathbb{F}_p((t))[\varphi]$- or $\mathbb{F}_p(t)[\varphi]$-module admit quantifier elimination in some natural language?

Rohwer works with predicates V_i that are interpreted by the sets of all elements of value $\geq i$. This gives less information than a binary predicate $P(x, y)$ interpreted by $vx \leq vy$ ("valuation divisibility").

Open Problem 11 What are the model theoretic properties of the elementary theory of $\mathbb{F}_p((t))$ as a valued $\mathbb{F}_p((t))[\varphi]$- or $\mathbb{F}_p(t)[\varphi]$-module in a language which includes a binary predicate for valuation divisibility?

Open Problem 12 What is the structure of extensions of valued $K[\varphi]$-modules? Can one prove Ax–Kochen–Ershov principles for valued $K[\varphi]$-modules?

Theorem 1 *Let F be an algebraic function field of transcendence degree 1 over a perfect field K of characteristic $p > 0$. If K is relatively algebraically closed in F, then there exists a Frobenius-closed basis for $F|K$, and F/K is a free $K[\varphi]$-module.*

Open Problem 13 Does Theorem 1 also hold for transcendence degree > 1?

Open Problem 14 Does Theorem 1 also hold if the assumption that K be perfect is replaced by the assumption that $F|K$ be separable?

E. Extremal valued fields.

Open Problem 15 Is every maximal field extremal? Is every henselian field with residue characteristic 0 extremal?

Open Problem 16 If a henselian field of characteristic p is extremal with respect to all p-polynomials in several variables, does this imply that it is extremal?

Open Problem 17 The following result indicates that additive polynomials are in some sense representative for the behaviour of values of images of polynomials in one variable:

Let (K,v) be a henselian field and $(a_\rho)_{\rho<\lambda}$ a pseudo Cauchy sequence in K without a limit in K. Pick a polynomial f of minimal degree such that the value $vf(a_\rho)$ is not ultimately fixed. Then there is an additive polynomial $\mathcal{A}\in K[X]$ such that for all large enough ρ,

$$v(f(a_\rho) - \mathcal{A}(a_\rho)) > vf(a_\rho)$$

(which in particular implies that $vf(a_\rho) = v\mathcal{A}(a_\rho)$).

Is something similar also true for polynomials in several variables?

Open Problem 18 Do we obtain a complete axiom system for $\mathbb{F}_p((t))$ if we add extremality to the axiom system stated above?

F. Additive polynomials and resolution.

Open Problem 19 What do the properties of additive polynomials mean in algebraic geometry, and in particular for resolution of singularities?

Open Problem 20 What is the behaviour of independent and dependent Artin–Schreier defect extensions when it comes to various forms of "relative resolution" (in the sense of Cutkosky – Piltant [C–P])?

Open Problem 21 Valuation theoretically speaking, independent Artin–Schreier defect extensions cannot be transformed into purely inseparable defect extensions. What information is lost when we do it "geometrically"?

Open Problem 22 What are the (Artin–Schreier) defect extensions that can appear over rational function fields, depending on the transcendence degree and on the value group?

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