Second International Conference and Workshop on Valuation Theory.
Segovia and El Escorial. Spain. July 18-29, 2011.

Ultrametrics and Implicit Function Theorems

Franz-Viktor Kuhlmann

The usual Implicit Function Theorem over a valued field is closely related to the multi-dimensional Hensel's Lemma. Both hold if and only if the field is henselian, that is, admits a unique extension of its valuation to its algebraic closure. Hensel's Lemma was originally proved by Hensel for the fields of p-adic numbers. The common proof used here is the valuation theoretical version of the Newton algorithm. But if one attempts to generalize the proof to other valued fields, for instance, to any maximal field, one may be forced to use transfinite induction, and the proof becomes much less elegant and effective. This fact, as well as the desire to formulate a Hensel's Lemma for other valued structures than valued fields, triggered the search for a more universal underlying principle. It turned out that this principle does not need addition or multiplication, it can already be formulated on the level of ultrametric spaces. (Every valuation induces an ultrametric.) This paves the way to proving Hensel's Lemmas not only in the case of less algebraic structure, but also for valued fields with more structure, such as derivations or automorphisms.

The principle I formulated in [1] is a "Main Theorem" which gives a criterion for the surjectivity of so-called *immediate mappings* on ultrametric spaces. The concept of immediate mappings is a natural generalization of the notion of immediate extensions of valued fields. The Main Theorem provides a uniform tool to prove all sorts of Hensel's Lemmas and generalized Hensel's Lemmas, including an easy proof of the multi-dimensional Hensel's Lemma on maximal fields, from where it can be pulled down to any henselian field.

But the Main Theorem also allows us to push the limits further: it can be used to prove an infinite-dimensional Implicit Function Theorem over suitable valued rings. The need for such infinite-dimensional versions has come up in the work of Bernard Teissier on local uniformization.

I will give a short introduction to ultrametric spaces, immediate mappings and the Main Theorem. Then I will describe its applications to the proofs of the multi-dimensional Hensel's Lemma and of an infinite-dimensional Implicit Function Theorem.

My work has been inspired by work of Paulo Ribenboim and Sibylla Priess-Crampe, and the Main Theorem has close connections with their Fixed Point and Attractor Theorems.

Keywords: Ultrametric space, Hensel's Lemma, Implicit Function Theorem

Mathematics Subject Classification 2000: 12J10, 12J20, 13A18

References

[1] F.-V. Kuhlmann. Maps on Ultrametric Spaces, Hensel's Lemma, and Differential Equations Over Valued Fields. *Communications in Algebra* **39** (5), 1730–1776, 2011.

Department of Mathematics and Statistics
University of Saskatchewan
106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada
fvk@math.usask.ca